This unit (Differentiation Rules) has two parallel goals:

- to become familiar with the computational techniques to determine derivatives so you don't have to use the definition of derivative to calculate derivatives, and
- to understand how these computational techniques arise.

This handout focuses on the latter, and proves some of the more important differentiation techniques which are outlined in Section 3.1 and 3.2. You will have to reproduce some of these arguments on the unit test.

Basic Derivative Rules (using Leibniz notation)

- Derivative of a constant function $f(x) = c$:

$$\frac{d}{dx}[c] = 0$$

- Power Rule $f(x) = x^n$, n is *any real number* (very nice proof we will look at in Section 3.6):

$$\frac{d}{dx}[x^n] = nx^{(n-1)}$$

- The Constant Multiple Rule: if c is a constant and f is a differentiable function:

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

- The Sum Rule: if f and g are both differentiable:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

- The Difference Rule: if f and g are both differentiable:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

- The product rule: the product rule is **not** $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$!

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

- The quotient rule:

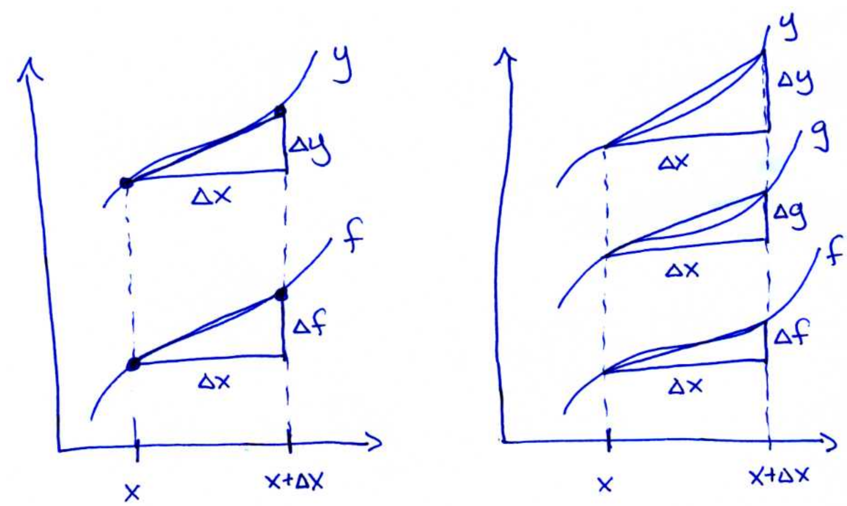
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g^2(x)} = \frac{\text{lo-de-hi minus hi-de-lo}}{\text{lo}^2}$$

- The Natural Exponential Function:

$$\frac{d}{dx}[e^x] = e^x$$

The Proofs

The key to all these proofs is a good schematic to begin. A schematic is not an accurate sketch (it can't be, since you don't know what f and g are!), but it will help you explain the process you are using to complete the proof. I've "stacked" y and f in my graph to make it easier to see the rectangles I've drawn. The functions y and f could cross or be below the x-axis, but the point of a schematic is to make the situation as simple as possible.



Use the one on the left if you only need f, the one on the right if you need an f and g.

These schematics show a few things. As $\Delta x \to 0$, the following is true:

$$\begin{aligned}
\Delta f &\to 0 \\
\Delta g &\to 0 \\
\Delta y &\to 0 \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= y' \\
\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} &= f' \\
\lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} &= g'
\end{aligned}$$

In all of the proofs, we will proceed as follows:

- Say the following: A small change in x (this is the Δx) leads to a small change in y (this is the Δy) and a small change in f (this is the Δf). Introduce this into your notation by replacing y with $y + \Delta y$ and f with $f + \Delta f$.
- Solve for Δy.
- Thinking about what you are trying to do, divide by Δx *in a smart way*.
- Take the limit as $\Delta x \to 0$, and use the limit laws and results from the schematic to simplify.

Practice will make the process clear. In each case, you are *building* $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

• Derivative of a constant function $y = c$. This does not require the schematic.

The slope of the line $y = c$ is always zero, since the tangent line is always horizontal. So $y' = 0$.

$$\frac{d}{dx}[c] = 0 \quad \text{(Leibniz notation)}$$

• The Constant Multiple Rule: $y = cf$, where c is a constant and f is a function of x. Note that we could have written this as $y = xf(x)$, but I am suppressing the functional dependence in all the proofs to improve clarity.

Your proof should begin with the schematic on the left on Page 2.

$$\begin{aligned}
y &= cf \text{ Write the definition of } y. \\
y + \Delta y &= c(f + \Delta f) \text{ Small change in } x \text{ introduces small changes in everything that depends on } x. \\
\Delta y &= cf + c\Delta f - y \text{ Solve for } \Delta y. \\
\Delta y &= cf + c\Delta f - cf \text{ Use definition of } y \text{ to simplify.} \\
\Delta y &= c\Delta f \\
\frac{\Delta y}{\Delta x} &= c\frac{\Delta f}{\Delta x} \text{ Divide by } \Delta x. \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= c \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \text{ Take limit as } \Delta x \to 0. \text{ Use limit laws.} \\
y' &= cf' \text{ Simplify by using results from your schematic. Done.} \\
\frac{d}{dx}[cf(x)] &= c\frac{d}{dx}f(x) \quad \text{(Leibniz notation)}
\end{aligned}$$

• The Sum Rule (Difference Rule is similar): $y = f + g$ where f and g both depend on x and are both differentiable:

Your proof should begin with the schematic on the right on Page 2.

$$\begin{aligned}
y &= f + g \\
y + \Delta y &= (f + \Delta f) + (g + \Delta g) \\
\Delta y &= (f + \Delta f) + (g + \Delta g) - y \\
\Delta y &= f + \Delta f + g + \Delta g - f - g \\
\Delta y &= \Delta f + \Delta g \\
\frac{\Delta y}{\Delta x} &= \frac{\Delta f}{\Delta x} + \frac{\Delta g}{\Delta x} \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} \\
y' &= f' + g' \\
\frac{d}{dx}[f(x) + g(x)] &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \quad \text{(Leibniz notation)}
\end{aligned}$$

• The Product Rule: $y = fg$ where f and g both depend on x and are both differentiable:

Your proof should begin with the schematic on the right on Page 2.

$$\begin{aligned}
y &= fg \\
y + \Delta y &= (f + \Delta f)(g + \Delta g) \\
\Delta y &= (f + \Delta f)(g + \Delta g) - y \\
&= fg + g\Delta f + f\Delta g + \Delta f \Delta g - y \\
&= y + g\Delta f + f\Delta g + \Delta f \Delta g - y \\
\text{So: } \Delta y &= g\Delta f + f\Delta g + \Delta f \Delta g \\
\text{Divide by } \Delta x\text{: } \frac{\Delta y}{\Delta x} &= g\frac{\Delta f}{\Delta x} + f\frac{\Delta g}{\Delta x} + \Delta f \frac{\Delta g}{\Delta x} \\
\text{Take Limit: } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= g \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + f \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} + \lim_{\Delta x \to 0} \Delta f \frac{\Delta g}{\Delta x} \\
y' &= gf' + fg' + \lim_{\Delta x \to 0} \Delta f \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} \\
y' &= gf' + fg' + (0)g' \\
y' &= gf' + fg' \\
\frac{d}{dx}[f(x)g(x)] &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \quad \text{(Leibniz notation)}
\end{aligned}$$

• The Quotient Rule: $y = f/g$ where f and g both depend on x and are both differentiable:

Your proof should begin with the schematic on the right on Page 2.

$$\begin{aligned}
y &= \frac{f}{g} \\
y + \Delta y &= \frac{f + \Delta f}{g + \Delta g} \\
\Delta y &= \frac{f + \Delta f}{g + \Delta g} - y \\
&= \frac{f + \Delta f}{g + \Delta g} - \frac{y(g + \Delta g)}{g + \Delta g} = \frac{f + \Delta f - y(g + \Delta g)}{g + \Delta g} = \frac{f + \Delta f - \frac{f}{g}(g + \Delta g)}{g + \Delta g} = \frac{f + \Delta f - f - f\frac{\Delta g}{g}}{g + \Delta g} \\
&= \frac{\Delta f - f\frac{\Delta g}{g}}{g + \Delta g} = \frac{g\Delta f - f\Delta g}{g(g + \Delta g)} \\
\frac{\Delta y}{\Delta x} &= \frac{g\frac{\Delta f}{\Delta x} - f\frac{\Delta g}{\Delta x}}{g(g + \Delta g)} \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \frac{g \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} - f \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}}{g(g + \lim_{\Delta x \to 0} \Delta g)} \\
y' &= \frac{gf' - fg'}{g^2} \\
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g^2(x)} \quad \text{(Leibniz notation)}
\end{aligned}$$

Definition of The Number e

Let $f(x) = a^x$, find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x f'(0)$$

The last bit comes from

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h}.$$

- The number e is the base (that is, the value of a) which has $f'(0) = 1$:

$$\begin{aligned} (e^x)' &= e^x \\ \frac{d}{dx}e^x &= e^x \quad \text{(Leibniz notation)} \end{aligned}$$

and we have the definition of the number e as:

e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$