1101 Calculus I Section 1.1

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set R.

The **range** R is the set of all possible values of $f(x)$, when x varies over the entire **domain** D.

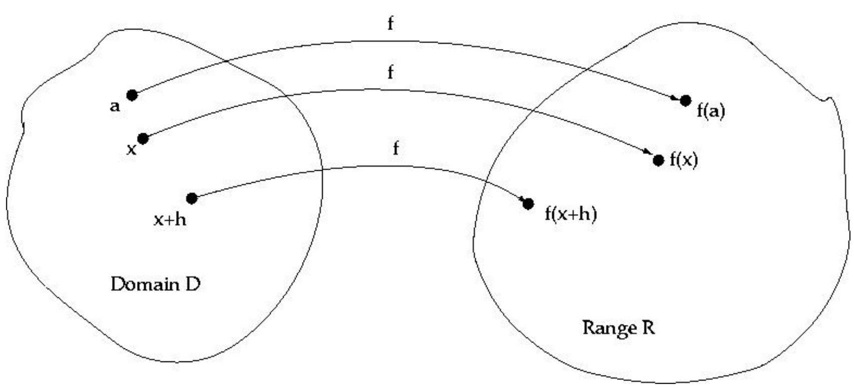
The functions we consider have the domain and range as real numbers, denoted $\mathbb{R}$.

The symbol (x in this case) which represents an arbitrary element in the domain of f is called the independent variable.

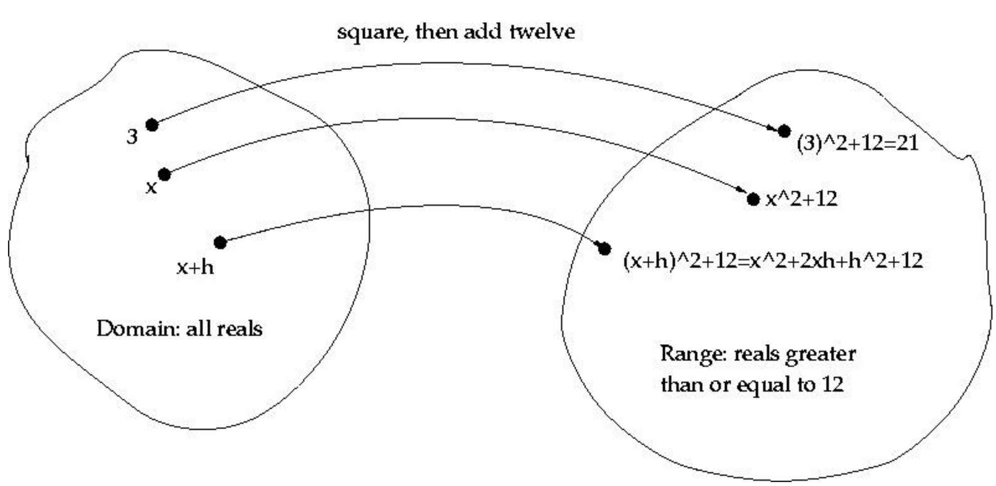
The symbol ($f(x)$ in this case) which represents an arbitrary element in the range of f is called the dependent variable (*dependent* because it depends on the value of x).

We often use $y = f(x)$ as dependent variable. This notation is called *Euler's function notation.* This is read as "y equals f of x". Note that this is not multiplication, that is, $f(x)$ does <u>not</u> mean f times x.

Arrow Diagram:



Arrow Diagram: Specific Function Here is an example of an arrow diagram for the function square, then add twelve.



Example Given $f(x) = x^2$, simplify the quantity $f(x+h) - f(x-h)$ as much as possible.

$$\begin{aligned} f(x+h) - f(x-h) &= (x+h)^2 - (x-h)^2 \\ &= (x^2 + h^2 + 2xh) - (x^2 + h^2 - 2xh) \\ &= x^2 + h^2 + 2xh - x^2 - h^2 + 2xh \\ &= 4xh \end{aligned}$$

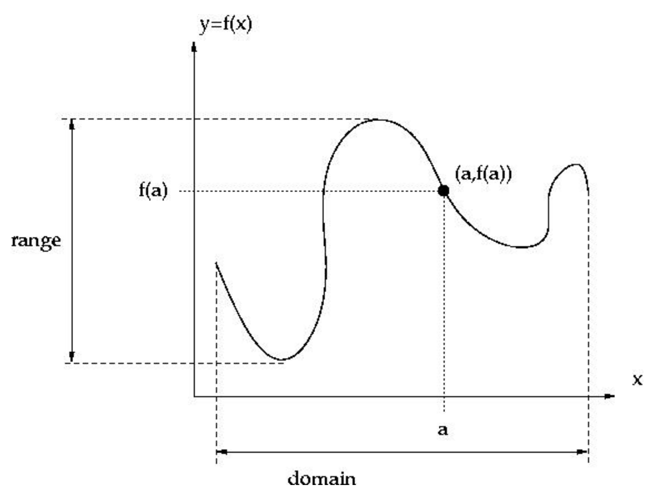
Graph A graph pictorially represents the relationship between ordered pairs, where the first element in the pair is the domain, the second element the range:

$$\{(x, f(x)) | x \in D\}$$

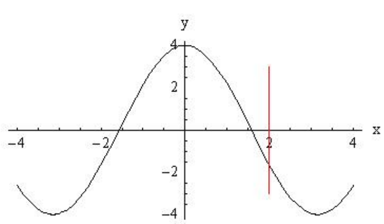
read: "ordered pair $(x, f(x))$ such that x is an element of D which is the domain."

The graph contains more information than the other descriptions.

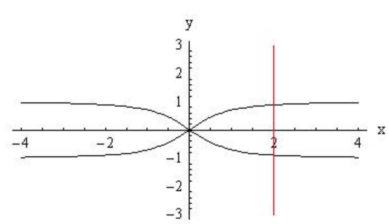
General Example:



Vertical Line Test A graph represents a function if every vertical line you can draw intersects the graph only once (this ensures we have exactly one element $f(x)$ for each x).



this graph represents a function



this graph does not represent a function

More on Domain and Range

Given $y = f(x)$, the values of x that can go into $f(x)$ and yield an output which is a real number form the domain. All the possible y's that come out form the range.

Example Find the domain and range of $h(x) = \dfrac{\sqrt{4-x^2}}{x-5}$.

We cannot have division by zero, so we want to see where the denominator is zero and exclude that value of x from the domain:

$$\begin{aligned} x - 5 &= 0 \\ x &= 5 \end{aligned}$$

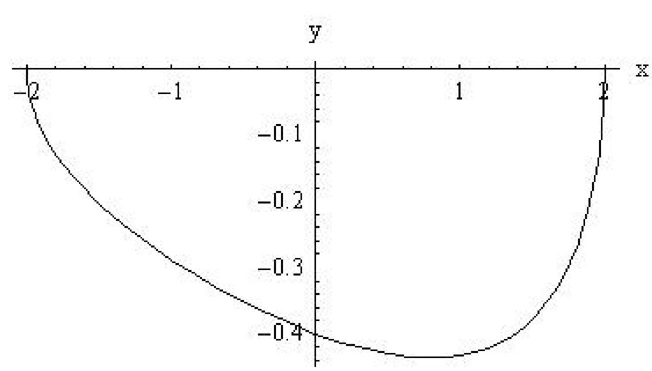
so $x = 5$ is not in the domain.

We also cannot take the square root of a negative number and get a result which is a real number. So we must have values of x for which

$$\begin{aligned} 4 - x^2 &\geq 0 \\ 4 &\geq x^2 \\ x^2 &\leq 4 \\ -\sqrt{4} \leq \sqrt{x^2} &\leq \sqrt{4} \\ -2 \leq x &\leq 2 \end{aligned}$$

This means the domain of the function $h(x)$ is $-2 \leq x \leq 2$, or $x \in [-2, 2]$. The point $x = 5$ is excluded, but that is already contained in the restriction based on the square root.

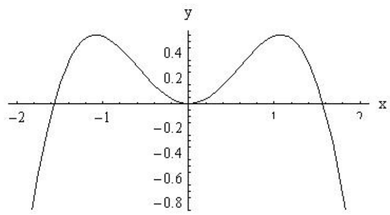
The range is all possible output values. This is usually more complicated to figure out than the domain, but easy to find if we plot a graph:



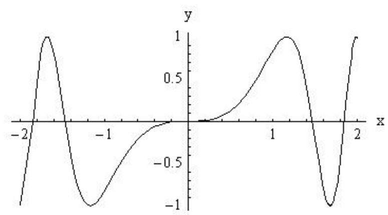
From the graph, we estimate the range to be $y \in [-0.44, 0]$. To get the range precisely we could use ideas from calculus, which we will learn later.

Symmetry

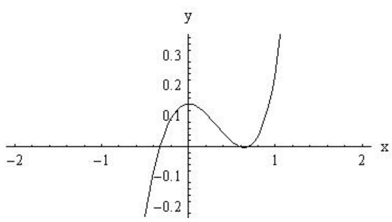
Even functions satisfy $f(-x) = f(x)$. Geometrically this means the function is symmetric about the y-axis.



Odd functions satisfy $f(-x) = -f(x)$. Geometrically this means the function is symmetric if we rotate 180 degrees about the origin.



NOTE: A function can be either even, or odd, or neither!



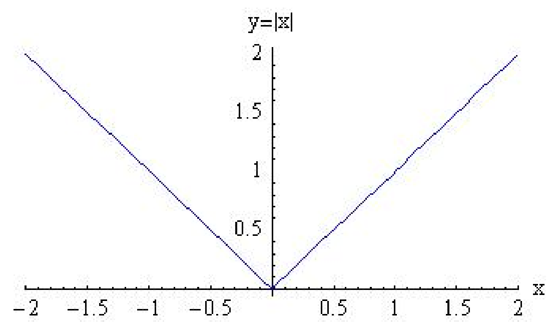
Example Determine whether the function $g(x) = \dfrac{x^3 - x}{x^3 + x}$ is even, odd, or neither. Use the algebraic technique to determine if a function is even or odd, rather than attempting to sketch the function.

$$
\begin{aligned}
g(-x) &= \frac{(-x)^3 - (-x)}{(-x)^3 + (-x)} \\
&= \frac{-x^3 + x}{-x^3 - x} \\
&= \frac{-(x^3 - x)}{-(x^3 + x)} \\
&= \frac{(x^3 - x)}{(x^3 + x)} = g(x)
\end{aligned}
$$

Since $g(-x) = g(x)$, the function g is even.

Piecewise Defined Functions are defined by different formulas for different parts of their domains.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$



$$h(x) = \begin{cases} x^3 \text{ if } x \leq 1 \\ -x^2 \text{ if } x > 1 \end{cases}$$