Examples

Example 1 Find the equation of the tangent line to the parabola $y = x^2 - x - 4$ at the point $P(1, -4)$.

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Example 4 Find $f'(a)$ if $f(x) = \sqrt{3x + 1}$.

Example 5 A particle moves along a straight line with equation of motion $s = f(t) = 2t^3 - t$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

Example 6 Find an equation of the tangent line to the function $y = 5/(x - 2)$ at the point $(1, -5)$.

Example 7 The position of a particle is given by $s(t) = \sqrt{t^2 + 1}$ where s is measured in meters and t is measured in seconds. What is the instantaneous velocity of the particle when $t = 1$ second?

Solutions

Example 1 Find the equation of the tangent line to the parabola $y = x^2 - x - 4$ at the point $P(1, -4)$.

Here we have $a = 1$ and $f(x) = x^2 - x - 4$, so the slope is:

$$\begin{aligned}
m &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x \to 1} \frac{(x^2 - x - 4) - (-4)}{x - 1} \\
&= \lim_{x \to 1} \frac{x(x - 1)}{x - 1} \\
&= \lim_{x \to 1} (x) \\
&= 1
\end{aligned}$$

Use the point slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

with $m = 1$ and $(x_1, y_1) = P = (1, -4)$ we have the equation for the tangent line:

$$y - (-4) = 1(x - 1) \text{ or } y = x - 5$$

The slope of the tangent line to a curve at a point is sometimes referred to as the slope of the curve at the point. This is because the tangent line approximates the curve at the point.

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f[x_] = x^2 - x - 4
tangent[x_] = x - 5
Plot[{f[x], tangent[x]}, {x, -5, 5}]
Plot[{f[x], tangent[x]}, {x, 0.5, 1.5}]
Plot[{f[x], tangent[x]}, {x, 0.9, 1.1}]
```

The slope could also be calculated using the alternate formula:

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{\big((a+h)^2-(a+h)-4\big)-\big(a^2-a-4\big)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}(a^2+h^2+2ah-a-h-4-a^2+a+4) \\
&= \lim_{h\to 0} \frac{1}{h}(h^2+2ah-h) \\
&= \lim_{h\to 0} \frac{h}{h}(h+2a-1) \\
&= \lim_{h\to 0} (h+2a-1) \\
&= 2a-1
\end{aligned}$$

Since we are interested in $a = 1$, the slope at $(1, -4)$ is $m = 2(1) - 1 = 1$.

Example 2 Find the derivative of $f(x) = x^2 - 8x + 9$ at $x = a$.

This can be solved using either of the two forms for derivative. The first is in your text:

$$\begin{aligned}
f(a) &= a^2-8a+9 \\
f(a+h) &= (a+h)^2-8(a+h)+9 \\
&= a^2+h^2+2ah-8a-8h+9 \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{a^2+h^2+2ah-8a-8h+9-a^2+8a-9}{h} \\
&= \lim_{h\to 0} \frac{1}{h}(h^2+2ah-8h) \\
&= \lim_{h\to 0} (h+2a-8) = 2a-8
\end{aligned}$$

The second solution would be:

$$\begin{aligned}
f(x) &= x^2-8x+9 \\
f(a) &= a^2-8a+9 \\
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{(x^2-8x+9)-(a^2-8a+9)}{x-a} \\
&= \lim_{x\to a} \frac{x^2-8x+9-a^2+8a-9}{x-a}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x\to a} \frac{x^2 - 8x - a^2 + 8a}{x-a} \\
&= \lim_{x\to a} \frac{x^2 - a^2 - 8(x-a)}{x-a} \\
&= \lim_{x\to a} \frac{(x+a)(x-a) - 8(x-a)}{x-a} \\
&= \lim_{x\to a} \frac{((x+a) - 8)(x-a)}{x-a} \\
&= \lim_{x\to a} ((x+a) - 8) \\
&= (a + a - 8) = 2a - 8
\end{aligned}$$

Example 2a Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(5, -6)$.

Let $f(x) = y$. Then, $f'(a) = 2a - 8$ is the slope of the tangent line at $x = a$. Here, $a = 5$. $m = f'(5) = 2(5) - 8 = 2$. The point-slope equation for a line is

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - (-6) &= 2(x - 16) \\
y &= 2x - 16
\end{aligned}$$

is the equation of the tangent line to $f(x)$ at the point $(5, -6)$.

In *Mathematica*:

```
Plot[{x^2 - 8x + 9, 2x-16}, {x, -3, 6}]
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Example 3 The position of a particle is given by the equation of motion $s = f(t) = 1/(1+t)$, where t is in seconds and s is in meters. Find the velocity and speed of the particle at $t = 2$ seconds.

I will work in general at $t = a$, and then substitute $a = 2$ at the end.

$$\begin{aligned}
f(a) &= \frac{1}{1+a} \\
f(a+h) &= \frac{1}{1+a+h} \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{1+a+h} - \frac{1}{1+a}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{1+a+h} - \frac{1}{1+a}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1+a-(1+a+h)}{(1+a+h)(1+a)}\right) \\
&= \lim_{h\to 0} \left(\frac{-1}{(1+a+h)(1+a)}\right) \\
&= \frac{-1}{(1+a)(1+a)} = \frac{-1}{(1+a)^2}
\end{aligned}$$

After 2 seconds, the velocity is therefore $f'(2) = -1/9$ m/s. The speed is the the absolute value of the velocity, so the speed is $|f'(2)| = 1/9$ m/s.

Example 4 Find $f'(a)$ if $f(x) = \sqrt{3x+1}$.

$$\begin{aligned}
f(a) &= \sqrt{3x+1} \\
f(a+h) &= \sqrt{3(a+h)+1} \\
&= \sqrt{3a+3h+1} \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3a+3h+1}-\sqrt{3a+1}}{h} \quad \text{Direct substitution yields indeterminant quotient} \\
&= \lim_{h\to 0} \frac{\sqrt{3a+3h+1}-\sqrt{3a+1}}{h} \cdot \left(\frac{\sqrt{3a+3h+1}+\sqrt{3a+1}}{\sqrt{3a+3h+1}+\sqrt{3a+1}}\right) \quad \text{rationalize the numerator} \\
&= \lim_{h\to 0} \frac{(3a+3h+1)-(3a+1)}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})} \\
&= \lim_{h\to 0} \frac{3a+3h+1-3a-1}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}} \\
&= \frac{3}{\sqrt{3a+3(0)+1}+\sqrt{3a+1}} \quad \text{Direct substitution now works} \\
&= \frac{3}{2\sqrt{3a+1}}
\end{aligned}$$

Example 5 A particle moves along a straight line with equation of motion $s = f(t) = 2t^3 - t$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

The velocity is equal to the derivative of the position.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
f(t) &= 2t^3 - t \\
f(a) &= 2a^3 - a \\
f(a+h) &= 2(a+h)^3 - (a+h) \\
&= 2(a^3+3a^2h+3ah^2+h^3) - a - h \\
&= 2a^3+6a^2h+6ah^2+2h^3 - a - h \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{(2a^3+6a^2h+6ah^2+2h^3-a-h)-(2a^3-a)}{h} \quad \text{Direct substitution won't work} \\
&= \lim_{h\to 0} \frac{1}{h}[2a^3+6a^2h+6ah^2+2h^3-a-h-2a^3+a] \\
&= \lim_{h\to 0} \frac{1}{h}[6a^2h+6ah^2+2h^3-h]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h\to 0}\frac{1}{h}[h(6a^2+6ah+2h^2-1)] \\
&= \lim_{h\to 0}(6a^2+6ah+2h^2-1) \quad \text{Direct substitution now works!} \\
&= 6a^2+6a(0)+2(0)^2-1 \\
&= 6a^2-1
\end{aligned}$$

The velocity when $t = 2$ s is $v(a) = f'(a) = 6a^2 - 1$. When $t = 2$, the velocity is $6(2)^2 - 1 = 23$ m/s.

Example 6 Find an equation of the tangent line to the function $y = 5/(x-2)$ at the point $(1, -5)$.

Let $f(x) = 5/(x-2)$. Then the slope of the tangent at $(a, f(a))$ is

$$\begin{aligned}
m &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0}\frac{\frac{5}{(a+h)-2}-\frac{5}{a-2}}{h} = \lim_{h\to 0}\frac{\left(\frac{5}{a+h-2}-\frac{5}{a-2}\right)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\left(\frac{5}{a+h-2}\right)\cdot\left(\frac{a-2}{a-2}\right)-\left(\frac{5}{a-2}\right)\cdot\left(\frac{a+h-2}{a+h-2}\right)\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{5(a-2)-5(a+h-2)}{(a-2)(a+h-2)}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{5a-10-5a-5h+10)}{(a-2)(a+h-2)}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{-5h}{(a-2)(a+h-2)}\right) \\
&= -\lim_{h\to 0}\frac{h}{h}\left(\frac{5}{(a-2)(a+h-2)}\right) \\
&= -\lim_{h\to 0}\left(\frac{5}{(a-2)(a+h-2)}\right) \\
&= -\left(\frac{5}{(a-2)(a+0-2)}\right) \\
&= -\frac{5}{(a-2)^2}
\end{aligned}$$

We are interested in $a = 1$, so the slope is $m = -5$.

Use the point slope form of the equation of a line: $y - y_1 = m(x - x_1)$.
Therefore, the equation of the tangent line at the point (1,-5) is $y - (-5) = -5(x-1) \longrightarrow y = -5x$.

Example 7 The position of a particle is given by $s(t) = \sqrt{t^2+1}$ where s is measured in meters and t is measured in seconds. What is the instantaneous velocity of the particle when $t = 1$ second?

The instantaneous velocity when $t = a$ seconds is given by:

$$\begin{aligned}
v(a) &= \lim_{h\to 0}\frac{s(a+h)-s(a)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{(a+h)^2+1}-\sqrt{a^2+1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{a^2+h^2+2ah+1}-\sqrt{a^2+1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{a^2+h^2+2ah+1}-\sqrt{a^2+1}}{h}\cdot\frac{\sqrt{a^2+h^2+2ah+1}+\sqrt{a^2+1}}{\sqrt{a^2+h^2+2ah+1}+\sqrt{a^2+1}} \\
&= \lim_{h\to 0}\frac{(a^2+h^2+2ah+1)-(a^2+1)}{h(\sqrt{a^2+h^2+2ah+1}+\sqrt{a^2+1})} \\
&= \lim_{h\to 0}\frac{h^2+2ah}{h(\sqrt{a^2+h^2+2ah+1}+\sqrt{a^2+1})} \\
&= \lim_{h\to 0}\frac{h(h+2a)}{h(\sqrt{a^2+h^2+2ah+1}+\sqrt{a^2+1})} \\
&= \lim_{h\to 0}\frac{(h+2a)}{(\sqrt{a^2+h^2+2ah+1}+\sqrt{a^2+1})} \\
&= \frac{(0+2a)}{(\sqrt{a^2+0^2+2a(0)+1}+\sqrt{a^2+1})} \\
&= \frac{2a}{2\sqrt{a^2+1}} = \frac{a}{\sqrt{a^2+1}}
\end{aligned}$$

At $a = 1$ second, $v(1) = \dfrac{1}{\sqrt{2}}$ m/s. The units come from the definition.