1101 Calculus I 3.9 Related Rates

Related Rates in General

Related rates means *related rates of change*, and since rates of changes are derivatives, related rates really means *related derivatives.*

The only way to learn how to solve related rates problems is to practice.

The procedure to solve a related rates problem:

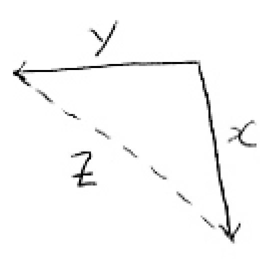
1. Write down the rate which is Given.
2. Write down the rate which is Unknown.
3. Write down your notation and draw a diagram.
4. Find a formula connecting the the quantities you listed in your Notation. There should be no derivatives in this relationship.
   (a) If necessary, use geometry to eliminate a variable from your formula.
5. Implicitly differentiate the formula to get rates of change involved. If you end up with more than one unknown rate of change, you might have to eliminate a variable using geometry (as mentioned in the previous step).
6. Solve for the Unknown Rate.
7. Substitute values to determine the Unknown Rate.
8. Write a concluding sentence.

You may vary from this procedure and still produce excellent solutions. Sometimes it helps to draw the diagram first.

Understand the examples in the text.

Example Two cars start moving from the same point. One travels south at 25 mph, and the other travels west at 15 mph. As what rate is the distance between the cars increasing 2 hours later?

1. Given: The rate of change of the distance (or velocity) of a car travelling south and a car travelling west.
2. Unknown: The rate of change of the distance between the two cars.
3. Diagram:

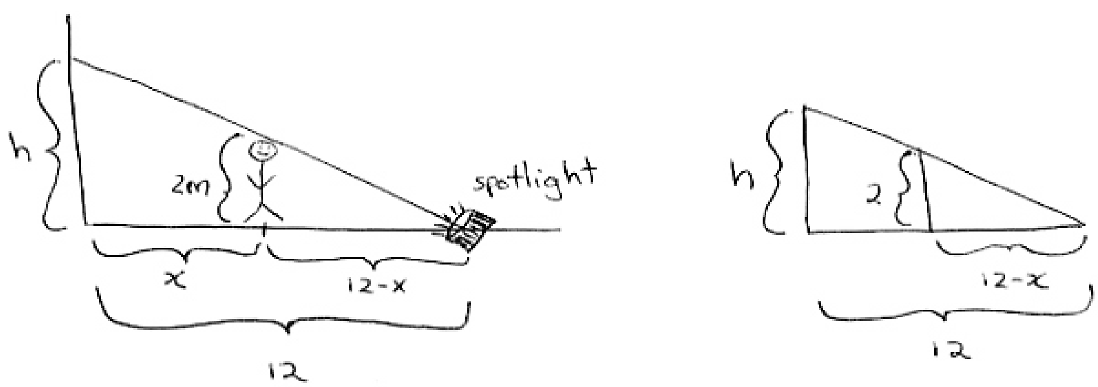


Notation:
x is the position of the car moving south.
y is the position of the car moving west.
$dx/dt = 25$ mph.
$dy/dt = 15$ mph.
we want to find dz/dt.

4. The formula relating the quantities is (from the diagram): $x^2 + y^2 = z^2$. Nothing needs to be eliminated.
5. Implicitly differentiate to introduce derivatives.
$$\begin{aligned} z^2 &= x^2 + y^2 \\ \frac{d}{dt}[z^2] &= \frac{d}{dt}[x^2 + y^2] \\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \end{aligned}$$
6. Solve for the Unknown Rate.
$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$
7. Substitute values to determine the Unknown Rate.
After 2 hours:
$x = 25$ mph (2 hrs) $= 50$ miles
$y = 15$ mph (2 hrs) $= 30$ miles
$z = \sqrt{x^2 + y^2} = \sqrt{50^2 + 30^2} = 10\sqrt{34}$ miles.
$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \\ &= \frac{1}{10\sqrt{34}}\left((50)(25) + (30)(15)\right) \\ &= 5\sqrt{34} \end{aligned}$$
8. After two hours, the distance between the two cars is increasing at a rate of $5\sqrt{34} \sim 29.15$ mph.

Example A spotlight on the ground shines on a wall 12m away. A man 2m tall walks from the spotlight towards the building at a speed of 1.6m/s. How fast is the length of his shadow on the building decreasing when he is 4m from the building?

1. We are Given the speed of the man.
2. We want to find the rate of change of the length of his shadow.
3. Diagram:



Notation:
x is the man's distance from the wall.
h is the height of the shadow.
The height of the man is 12m.
$dx/dt = -1.6$ m/s (negative since it is decreasing).
dh/dt is what we want to find.

4. Similar triangles:

$$\frac{h}{12} = \frac{2}{12-x}$$

$$h = \frac{24}{12-x}$$

Nothing needs to be eliminated.

5. Implicitly differentiate the formula to get rates of change involved.

$$\begin{aligned}\frac{d}{dt}[h] &= \frac{d}{dt}[\frac{24}{12-x}] \\ \frac{dh}{dt} &= 24\frac{d}{dt}[(12-x)^{-1}] \\ &= 24(-1)(12-x)^{-2}(-1)\frac{dx}{dt} \\ &= \frac{24}{(12-x)^2}\frac{dx}{dt}\end{aligned}$$

6. Solve for the Unknown Rate.
Done

7. Substitute values to determine the Unknown Rate.

$$\frac{dh}{dt} = \frac{24}{(12-x)^2}\frac{dx}{dt} = \frac{24}{(12-4)^2}(-1.6) = -0.6$$

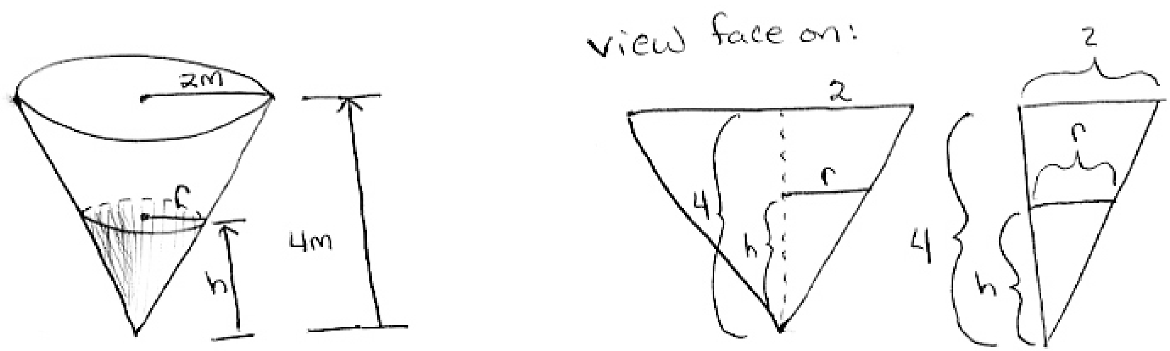
8. The size of the man's shadow is decreasing at a rate of 0.6 m/s when he is 4m from the wall.

Example A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at the rate of 2 m^3/min, find the rate at which the water is rising when the water is 3m deep.

Given: rate at which the volume of water is being increased in the tank.

Unknown: Rate at which the height of water is changing.

Diagram:



Notation:
Let V be the volume of water.
Let h be the height of water.
Let r be the radius of water.
$\frac{dV}{dt} = 2$ m^3/min, $\frac{dh}{dt}$ is unknown.

The formula relating the quantities in my Notation is:

$$V = \frac{1}{3}\pi r^2 h$$

Eliminate r, since we do not know r, or dr/dt! Use similar triangles relations:

$$\frac{h}{r} = \frac{4}{2} \rightarrow r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

Implicitly differentiate the formula to get derivatives involved. The quantities are functions of time, so differentiate with respect to t.

$$\frac{d}{dt}[V = \frac{1}{12}\pi h^3]$$
$$\frac{dV}{dt} = \frac{1}{12}\pi(3h^2)\frac{dh}{dt}$$

Solve for the Unknown Rate.

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substitute values to determine the Unknown Rate.

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}(2) = \frac{8}{9\pi} \text{ m/min}$$

The rate at which the water is rising when $h = 3$ m is $dh/dt = 8/(9\pi)$ m/min.