Note: It is generally alright to reuse letters (y, f, u, w, etc.) in your solution to a problem as long as they weren't part of the original question. For example, in Example 6, I reuse y, f, u (they stand for different things at different times in my solution) since they are quantities *I* introduced in my solution. However, in Example 6, y was part of the original question, so I do not reuse that variable in my solution. The exception to this is that you can reuse y if your answer is an equation of a tangent line!

Note on Trigonometric Functions When dealing with trig functions, we prefer to work in radians rather than degrees. Recall that to switch between the two measures, we can think that a circle is swept out when the angle goes through 360 degrees, or 2π radians: 2π radians $= 360$ degrees . The reason for this is exemplified in the following question.

On the test which is coming up you will have access to *Mathematica* to check derivatives and solve equations, but there will still be problems which will require you to work out derivatives by hand, and you will still need to show your work to get full credit. Remember, *Mathematica* does not do our thinking for us!

1. Example 3.5.77 Use the chain rule to show that if θ is measured in degrees, and x is the same angle measured in radians, then

$$\frac{d}{dx}\sin\theta = \frac{\pi}{180}\cos\theta$$

Solution Recall, if x is measured in radians, we have

$$\frac{d}{dx}\sin x = \cos x$$

which is a simpler formula. The relation between radians and degrees is $\theta^o = \pi x/180$ radians.

$$\frac{d}{dx}\sin\theta = \frac{d}{d\theta}[\sin\theta]\frac{d\theta}{dx} = \cos\theta\frac{\pi}{180}$$

2. Example Find the equation of the tangent line to the curve $y = x + \cos x$ at $(0, 1)$.

Solution Statements:
The slope of the tangent line is the derivative of the function.
We want the equation of the tangent line, so our answer will look like $y - y_0 = m(x - x_0)$.
The point we are interested in is $(x_0, y_0) = (0, 1)$, which has $x = 0$.
We want to find the derivative $f'(0) = m$.
We need to define $f(x) = x + \cos x$.
Our answer will look like $y - 1 = f'(0)(x - 0)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x + \cos x] \\ &= 1 - \sin[x] \\ f'(0) &= 1 - \sin 0 = 1 \end{aligned}$$

The equation of the tangent line to the curve at (0,1) is

$$\begin{aligned} y - 1 &= f'(0)(x - 0) \\ y - 1 &= (1)x \\ y &= x + 1 \end{aligned}$$

3. Example Find the derivative of $g(t) = \dfrac{\sin^2 t}{\cos t}$.

Solution

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left[\frac{\sin^2 t}{\cos t}\right] \\ &= \frac{\cos t \frac{d}{dt}[\sin^2 t] - \sin^2 t \frac{d}{dt}[\cos t]}{\cos^2 t} \quad \text{quotient rule} \end{aligned}$$

The derivative of $\sin^2 t$ will require the chain rule.

$$y = \sin^2 t \text{ decomposition: } \begin{aligned} y &= u^2 \\ u &= \sin t \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}[\sin^2 t] &= \frac{dy}{dt} \\ &= \frac{dy}{du} \cdot \frac{du}{dt} \quad \text{chain rule} \\ &= (2u) \cdot (\cos t) \\ &= 2 \sin t \cos t \end{aligned}$$

$$\begin{aligned} g'(t) &= \frac{\cos t (2 \sin t \cos t) - \sin^2 t[-\sin t]}{\cos^2 t} \\ &= \frac{2 \sin t \cos^2 t + \sin^3 t}{\cos^2 t} \\ &= 2 \sin t + \sin t \tan^2 t \end{aligned}$$

4. Example Use the chain rule to prove that the derivative of an even function is an odd function.

Solution An even function will satisfy the equation:

$$\begin{aligned} f(x) &= f(-x) \quad \text{differentiate this equation} \\ \frac{d}{dx} f(x) &= \frac{d}{dx} f(-x) \\ f'(x) &= \frac{d}{dx} f(u), \qquad u = -x \\ &= \frac{d}{du} f(u) \cdot \frac{du}{dx} \quad \text{chain rule} \\ &= f'(u) \cdot (-1) \\ &= -f'(-x) \end{aligned}$$

since $f'(x) = -f'(-x)$, $f'(x)$ is odd! Much easier than our previous proof of this result using the definition of derivative (see Homework 2.9.47).

5. Example 3.5.75 If n is a positive integer, prove that

$$\frac{d}{dx}[\sin^n x \cos(nx)] = n \sin^{n-1} x \cos((n+1)x).$$

Solution

$$\frac{d}{dx}[\sin^n x \cos(nx)] = \frac{d}{dx}[\sin^n x] \cos(nx) + \sin^n x \frac{d}{dx}[\cos(nx)] \quad \text{product rule}$$

We need to use the chain rule to do the two derivatives. Let's do 'em!

$$y = \sin^n x \text{ decomposition: } \begin{aligned} f &= u^n \\ u &= \sin x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}[\sin^n x] &= \frac{d}{dx} f(u(x)) \\ &= \frac{df}{du} \cdot \frac{du}{dx} \quad \text{chain rule} \\ &= nu^{n-1} \cdot \cos x \\ &= n \sin^{n-1} x \cos x \end{aligned}$$

$$y = \cos(nx) \text{ decomposition: } \begin{aligned} f &= \cos u \\ u &= nx \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}[\cos(nx)] &= \frac{d}{dx} f(u(x)) \\ &= \frac{df}{du} \cdot \frac{du}{dx} \quad \text{chain rule} \\ &= (-\sin u) \cdot n \\ &= -n \sin(nx) \end{aligned}$$

Now we substitute back:

$$\begin{aligned} \frac{d}{dx}[\sin^n x \cos(nx)] &= \frac{d}{dx}[\sin^n x] \cos(nx) + \sin^n x \frac{d}{dx}[\cos(nx)] \quad \text{product rule} \\ &= n \sin^{n-1} x \cos x \cos(nx) + \sin^n x(-n \sin(nx)) \\ &= n \left(\sin^{n-1} x \cos x \cos(nx) - \sin^n x \sin(nx) \right) \\ &= n \sin^{n-1} x \left(\cos x \cos(nx) - \sin x \sin(nx) \right) \end{aligned}$$

Use the trig identity $\cos a \cos b - \sin a \sin b = \cos(a+b)$ to rewrite $\cos x \cos(nx) - \sin x \sin(nx) = \cos[(n+1)x]$ and we arrive at the answer,

$$\frac{d}{dx}[\sin^n x \cos(nx)] = n \sin^{n-1} x \cos[(n+1)x]$$

6. Example 3.5.22 $y = e^{-5x}\cos 3x$, find $y'(x)$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[e^{-5x}\cos 3x]\\ &= \frac{d}{dx}[e^{-5x}]\cos 3x + e^{-5x}\frac{d}{dx}[\cos 3x] \quad \text{product rule}\end{aligned}$$

We must use the chain rule to do the two derivatives.

$$f = e^{-5x} \text{ decomposition: } \begin{aligned} f &= e^u\\ u &= -5x\end{aligned}$$

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du}\cdot\frac{du}{dx} \quad \text{chain rule}\\ &= e^u\cdot(-5)\\ &= -5e^{-5x}\end{aligned}$$

$$f = \cos 3x \text{ decomposition: } \begin{aligned} f &= \cos u\\ u &= 3x\end{aligned}$$

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du}\cdot\frac{du}{dx} \quad \text{chain rule}\\ &= -\sin u\cdot(3)\\ &= -3\sin 3x\end{aligned}$$

Now we can substitute back:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[e^{-5x}\cos 3x]\\ &= \frac{d}{dx}[e^{-5x}]\cos 3x + e^{-5x}\frac{d}{dx}[\cos 3x]\\ &= -5e^{-5x}\cos 3x + e^{-5x}(-3\sin 3x)\\ &= -e^{-5x}(5\cos 3x + 3\sin 3x)\end{aligned}$$

7. Example 3.5.10 $f(t) = \dfrac{1}{(t^2 - 2t - 5)^4}$, find $f'(t)$.

Solution

$$\begin{aligned} f'(t) &= \frac{(t^2-2t-5)^4 \dfrac{d}{dt}[1] - (1)\dfrac{d}{dt}\left[(t^2-2t-5)^4\right]}{(t^2-2t-5)^8} && \text{quotient rule} \\ &= -\frac{\dfrac{d}{dt}\left[(t^2-2t-5)^4\right]}{(t^2-2t-5)^8} && \text{constant rule} \end{aligned}$$

We need to use the chain rule to do this derivative:

$$y = (t^2-2t-5)^4 \text{ decomposition: } \begin{aligned} y &= u^4 \\ u &= t^2 - 2t - 5 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{du} \cdot \frac{du}{dt} && \text{chain rule} \\ &= (4u^3) \cdot (2t-2) \\ &= 4(2t-2)(t^2-2t-5)^3 \end{aligned}$$

Now we can substitute back:

$$\begin{aligned} f'(t) &= -\frac{\frac{d}{dt}\left[(t^2-2t-5)^4\right]}{(t^2-2t-5)^8} \\ &= -\frac{4(2t-2)(t^2-2t-5)^3}{(t^2-2t-5)^8} \\ &= -\frac{4(2t-2)}{(t^2-2t-5)^5} \end{aligned}$$

Alternate solution Rewrite the function as $f(t) = (t^2-2t-5)^{-4}$. $f'(t) = \dfrac{d}{dt}\left[(t^2-2t-5)^{-4}\right]$. We need to use the chain rule to do this derivative:

$$f = (t^2-2t-5)^{-4} \text{ decomposition: } \begin{aligned} f &= u^{-4} \\ u &= t^2 - 2t - 5 \end{aligned}$$

$$\begin{aligned} \frac{df}{dt} &= \frac{df}{du} \cdot \frac{du}{dt} && \text{chain rule} \\ &= (-4u^{-5}) \cdot (2t-2) \\ &= -4(2t-2)(t^2-2t-5)^{-5} \\ f'(t) = \frac{df}{dt} &= -\frac{4(2t-2)}{(t^2-2t-5)^5} \end{aligned}$$

8. Example For what values of x does the curve $y = x + \cos 2x$ have horizontal tangents?

Solution Statements:
The slope of the tangent line is the derivative of the function.
We want the values of x, let's call them $x = a$, for which the tangent is horizontal.
We need to solve $f'(a) = 0$ for the number a.
We need to define $f(x) = x + \cos 2x$.
Our answer will be the numbers a.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x + \cos 2x] \\
&= 1 + \frac{d}{dx}[\sin 2x] \\
&\quad \text{decompose: } u = 2x, \\
&= 1 + \frac{d}{dx}[\sin u] \\
&= 1 + \frac{d}{du}[\sin u] \cdot \frac{du}{dx} \quad \text{(chain rule)} \\
&= 1 + [-\cos u] \cdot [2] \\
&= 1 - 2\cos 2x \\
f'(a) = 0 &\longrightarrow 1 - 2\sin 2a = 0 \\
&\quad \sin 2a = \frac{1}{2}
\end{aligned}$$

This can be solved by noting that $\sin(\pi/6) = 1/2$. This is one of our special angles.
There are other solutions since sine is periodic with period 2π, $\sin(\pi/6 + 2n\pi) = 1/2$, n is an integer.
Therefore, $2a = \dfrac{\pi}{6} + 2n\pi$, n is an integer.
Therefore, $a = \dfrac{\pi}{12} + n\pi$, n is an integer.

Alternatively, you can solve this using *Mathematica*:

```
f[x_] = x + Cos[2x]
f'[x]
Solve[f'[x] == 0, x]
Reduce[f'[x] == 0, x]
```

9. Example $y = \sin(e^x)$. Find y'.

Solution Here, y is a function of x, so $y\prime = dy/dx$.

$$y = \sin(e^x) \text{ decomposition: } \begin{aligned} y &= \sin u \\ u &= e^x \end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{chain rule} \\
&= (\cos u) \cdot (e^x) \\
&= e^x \cos(e^x)
\end{aligned}$$

10. Example 3.5.63 The displacement of a particle on a vibrating string is given by $s = A\cos(\omega t + \delta)$. Find the velocity of the particle at time t. When is the velocity zero?

Solution Statements:
The velocity is the derivative of the position function.
The position function is given by $s = f(t)$.
The velocity will be $v(t) = f'(t)$.
We will need to use the chain rule to calculate the derivative.
Once we have the velocity, we can determine for what time it is zero by solving $v(t) = 0$ for t.

$$f(t) = A\cos(\omega t + \delta) \text{ decomposition: } \begin{aligned} f &= A\cos h \\ h &= \omega t + \delta \end{aligned}$$

$$\begin{aligned} v(t) = f'(t) &= \frac{df}{dt} \\ &= \frac{df}{dh} \cdot \frac{dh}{dt} \quad \text{chain rule} \\ &= \frac{d}{dh}[A\cos h]\frac{d}{dt}[\omega t + \delta] \\ &= A(-\sin h)(\omega) \\ &= -A\omega\sin(\omega t + \delta) \end{aligned}$$

The velocity is zero when

$$-A\omega\sin(\omega t + \delta) = 0$$

which occurs when $\omega t + \delta = n\pi$, n an integer, so $t = (n\pi - \delta)/\omega$.

11. Example Find the x-coordinates in $(-\pi, \pi)$ for which the curve $y = \sin(2x) - 2\sin x$ has a horizontal tangent line. *Solution requires use of Mathematica to solve an equation.*

Solution Statements:
Slope of the tangent line is the derivative of the function.
Our function will be $f(x) = \sin(2x) - 2\sin x$.
We will have to use the chain rule to determine the derivative $(\sin 2x)$.
If the tangent line is horizontal, then the slope is zero.
We want to find all the points a which satisfy $f'(a) = 0$.
Our answer will be the numbers a.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin(2x) - 2\sin x] \\ &= \frac{d}{dx}[\sin(2x)] - 2\frac{d}{dx}[\sin x] \\ \frac{d}{dx}[\sin(2x)] &= \frac{d}{dx}[\sin u], \;\; u = 2x \\ &= \frac{d}{du}[\sin u]\frac{du}{dx} \quad \text{chain rule} \\ &= \cos u \cdot (2) \end{aligned}$$

$$\begin{aligned} &= 2\cos(2x) \\ f'(x) &= 2\cos(2x) - 2\cos x \end{aligned}$$

Use *Mathematica* to solve for a in $f'(a) = 0$:

```
Solve[2 Cos[2a] - 2 Cos[a] == 0, x]
```

And we find that $a = 0, -2\pi/3, 2\pi/3$.

12. Example Given $f(x) = \dfrac{\sqrt{x^2+1}}{\sec x \sin x + e^{2x}}$, find $f'(x)$.

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{\sqrt{x^2+1}}{\sec x \sin x + e^{2x}}\right] \\ &= \frac{(\sec x \sin x + e^{2x})\frac{d}{dx}[\sqrt{x^2+1}] - (\sqrt{x^2+1})\frac{d}{dx}[\sec x \sin x + e^{2x}]}{(\sec x \sin x + e^{2x})^2} \quad \text{(quotient rule)} \end{aligned} \tag{1}$$

Let's pause to work out the two derivatives as an aside.

$$\begin{aligned} \frac{d}{dx}[\sqrt{x^2+1}] &= \frac{d}{dx}[\sqrt{u}], \quad u = x^2+1 \\ &= \frac{d}{du}[\sqrt{u}] \cdot \frac{du}{dx}, \quad \text{(chain rule)} \\ &= \frac{1}{2}u^{1/2-1} \cdot (2x), \\ &= \frac{1}{2\sqrt{u}} \cdot (2x), \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{d}{dx}[\sec x \sin x + e^{2x}] &= \frac{d}{dx}[\sec x \sin x] + \frac{d}{dx}[e^{2x}] \quad \text{(sum rule)} \\ &= \frac{d}{dx}[\sec x]\sin x + \sec x\frac{d}{dx}[\sin x] + \frac{d}{dx}[e^u], \quad u = 2x \\ &\quad \text{(first term: product rule; second term: set up for chain rule)} \\ &= [\sec x \tan x]\sin x + \sec x[\cos x] + \frac{d}{du}[e^u] \cdot \frac{du}{dx}, \quad \text{(chain rule)} \\ &= [\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}]\sin x + \sec x[\frac{1}{\sec x}] + [e^u] \cdot (2), \quad \text{(simplify)} \\ &= \tan^2 x + 1 + 2e^{2x}. \end{aligned} \tag{3}$$

Now, we can substitute Equations (2) and (3) into Equation (1).

$$f'(x) = \frac{(\sec x \sin x + e^{2x})\frac{d}{dx}[\sqrt{x^2+1}] - (\sqrt{x^2+1})\frac{d}{dx}[\sec x \sin x + e^{2x}]}{(\sec x \sin x + e^{2x})^2}$$

$$= \frac{(\sec x \sin x + e^{2x})\left(\frac{x}{\sqrt{x^2+1}}\right) - (\sqrt{x^2+1})[\tan^2 x + 1 + 2e^{2x}]}{(\sec x \sin x + e^{2x})^2}$$

If we need to, we can simplify this. However, we don't have to if all we wanted was the derivative.

If we want to compare with what *Mathematica* gives us, we need to simplify a bit. Use the following: $\tan^2 x + 1 = \sec^2 x$, and $\sec x \sin x = \tan x$:

$$\begin{aligned}
f'(x) &= \frac{(\tan x + e^{2x})\left(\frac{x}{\sqrt{x^2+1}}\right) - (\sqrt{x^2+1})[\sec^2 x + 2e^{2x}]}{(\tan x + e^{2x})^2} \\
&= \frac{(\tan x + e^{2x})\left(\frac{x}{\sqrt{x^2+1}}\right)}{(\tan x + e^{2x})^2} - \frac{(\sqrt{x^2+1})[\sec^2 x + 2e^{2x}]}{(\tan x + e^{2x})^2} \\
&= \frac{x}{\sqrt{x^2+1}(\tan x + e^{2x})} - \frac{(\sqrt{x^2+1})(\sec^2 x + 2e^{2x})}{(\tan x + e^{2x})^2}
\end{aligned}$$

13. Example Given $f(x) = \cos(\cos(\cos(\cos x)))$, find $f'(x)$.

Solution This is a test of our chain rule abilities. Let's decompose!

$s = \cos w$, $w = \cos v$, $v = \cos u$, $u = \cos x$.

Check we did the decomposition correctly:

$$\begin{aligned}
(s \circ w \circ v \circ u)(x) &= s\Big(w\Big(v\big(u(x)\big)\Big)\Big) \\
&= s\Big(w\Big(v\big(\cos x\big)\Big)\Big) \\
&= s\Big(w\Big(\cos\big(\cos x\big)\Big)\Big) \\
&= s\left(\cos(\cos(\cos x))\right) \\
&= \cos(\cos(\cos(\cos x))) \\
&= f(x)
\end{aligned}$$

Therefore, we can use the chain rule as follows.

$$\begin{aligned}
f'(x) &= \frac{ds}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{dw}[\cos w] \cdot \frac{d}{dv}[\cos v] \cdot \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[\cos x] \\
&= [-\sin w] \cdot [-\sin v] \cdot [-\sin u] \cdot [-\sin x] \\
&= [\sin(\cos v)] \cdot [\sin(\cos u)] \cdot [\sin \cos x] \cdot [\sin x] \\
&= [\sin(\cos(\cos u))] \cdot [\sin(\cos(\cos x))] \cdot [\sin(\cos x)] \cdot [\sin x] \\
&= [\sin(\cos(\cos(\cos x)))] \cdot [\sin(\cos(\cos x))] \cdot [\sin(\cos x)] \cdot [\sin x]
\end{aligned}$$