Example

$$\begin{aligned}
\int \frac{1+x}{1+x^2}\,dx &= \int \left[\frac{1}{1+x^2} + \frac{x}{1+x^2}\right]\,dx \\
&= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx \\
&= \arctan x + \int \frac{x}{1+x^2}\,dx + c_1 \qquad \text{Substitution: } u = 1+x^2, du = 2x\,dx \\
&= \arctan x + \int \frac{1}{u}\frac{du}{2} + c_1 \\
&= \arctan x + \frac{1}{2}\int \frac{du}{u} + c_1 \\
&= \arctan x + \frac{1}{2}\ln u + c_1 + c_2 \\
&= \arctan x + \frac{1}{2}\ln(1+x^2) + c
\end{aligned}$$

Example

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \qquad \text{Substitution: } u = \cos x, du = -\sin x\,dx \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + c \\
&= -\ln|\cos x| + c \\
&= \ln|\frac{1}{\cos x}| + c \\
&= \ln|\sec x| + c
\end{aligned}$$

Example

$$\begin{aligned}
\int \sqrt{x-1}\,dx &= \int \sqrt{u}\,du \qquad \text{Substitution: } u = x-1, du = dx \\
&= \frac{u^{3/2}}{3/2} + c \\
&= \frac{2}{3}(x-1)^{3/2} + c
\end{aligned}$$

Example

$$\begin{aligned}
\int \frac{(1+\sqrt{x})^9}{\sqrt{x}}\,dx &= 2\int \frac{(1+\sqrt{x})^9}{2\sqrt{x}}\,dx \qquad \text{Substitution: } u = 1+\sqrt{x}, du = \frac{1}{2\sqrt{x}}\,dx \\
&= 2\int u^9\,du \\
&= 2\frac{u^{10}}{10} + c \\
&= \frac{1}{5}(1+\sqrt{x})^{10} + c
\end{aligned}$$

Example

$$\begin{aligned}\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx &= \int \frac{1}{\sqrt{u}}\,du \qquad \text{Substitution: } u = 1+x+2x^2, du = (1+4x)\,dx\\ &= \int u^{-1/2}\,du\\ &= \frac{u^{1/2}}{1/2} + c\\ &= 2\sqrt{1+x+2x^2} + c\end{aligned}$$

Example

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{1}{u}\,du \qquad \text{Substitution: } u = \ln x, du = \frac{1}{x}\,dx\\ &= \ln|u| + c\\ &= \ln|\ln x| + c\end{aligned}$$

Try the last one on *Mathematica*, what do you notice? No absolute value bars!!
Where is this integral formula valid? Ans: $1 < x < \infty$.

Example

$$\begin{aligned}\int t^2\cos(1-t^3)\,dt &= \frac{1}{-3}\int \cos(1-t^3)(-3t^2)\,dt \qquad \text{Substitution: } u = 1-t^3, du = -3t^2\,dt\\ &= -\frac{1}{3}\int \cos u\,du\\ &= -\frac{1}{3}\sin u + c\\ &= -\frac{1}{3}\sin(1-t^3) + c\end{aligned}$$

Try the last one on *Mathematica*, what do you notice? The form looks different! They are the same.

Example

$$\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx$$

$$\text{Substitution: } \begin{array}{ll} u = 1+\frac{1}{x} & x = 1 \to u = 2\\ du = -\frac{1}{x^2}dx & x = 4 \to u = 5/4\end{array}$$

$$\begin{aligned}\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx &= -\int_2^{5/4}\sqrt{u}\,du\\ &= -\left.\frac{u^{3/2}}{3/2}\right|_2^{5/4}\\ &= -\frac{2}{3}\left[\left(\frac{5}{4}\right)^{3/2} - 2^{3/2}\right]\end{aligned}$$

Example

$$\int_0^3 \frac{dx}{2x+3}$$

Substitution: $\begin{array}{ll} u = 2x+3 & x = 0 \to u = 3 \\ du = 2dx & x = 3 \to u = 9 \end{array}$

$$\begin{aligned} \int_0^3 \frac{dx}{2x+3} &= \frac{1}{2}\int_3^9 \frac{du}{u} \\ &= \frac{1}{2}\ln|u|_3^9 \\ &= \frac{1}{2}[\ln 9 - \ln 3] = \frac{\ln 3}{2} \end{aligned}$$

Example

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$$

Substitution: $\begin{array}{ll} u = \ln x & x = e \to u = 1 \\ du = \frac{dx}{x} & x = e^4 \to u = 4 \end{array}$

$$\begin{aligned} \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} &= \int_1^4 \frac{du}{\sqrt{u}} \\ &= \int_1^4 u^{-1/2}\, du \\ &= \left.\frac{u^{1/2}}{1/2}\right|_1^4 \\ &= 2[\sqrt{4} - \sqrt{1}] = 2[2-1] = 2 \end{aligned}$$