This is a set of practice test problems for Chapter 2. This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words–remember, I am interested in the process you use to solve problems!

You should know the skills we reviewed in Chapter 1, for example the quadratic formula, logarithm laws, exponential laws, algebra techniques (ie. rationalizing, common denominator, expanding powers, etc.), properties of basic functions (ie. domain of logarithm, sketch of $y = x^3$, etc.), functional notation ($f(x) = x^2$, then $f(a+h) = (a+h)^2$), etc.

There will be five problems on the test. Most will involve more than one part. You will have 100 minutes to complete the test. You may not use *Mathematica* or calculators on this test.

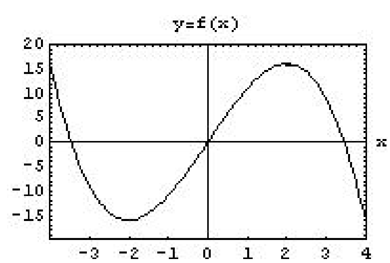
1. Draw an example of a function on the interval $[a, b]$ for which the Intermediate Value Theorem does not apply. Make sure your sketch is clearly labelled, and you explain in words why the Intermediate Value Theorem does apply to your function.

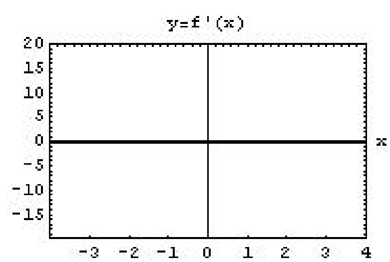
2. Sketch a function g which has the following properties: $\lim_{x\to -2} g(x) = \infty$, g has a removable discontinuity at $x = 1$, and $g'(0) = 1$.

3. Prove that if f and g are continuous at a, then $f - g$ is also continuous at a.

4. Given $f(x) = \sqrt{x^2 - 1}$, find $f'(x)$ using the definition of derivative.

5. You are given a sketch of the function f below. Use it to sketch the graph of the derivative f' in the coordinate system provided. Give a detailed explanation about why you drew the curve f' the way you did.





6. Draw a well labelled sketch, and give a short explanation of why the quantity $\lim_{h\to 0} \dfrac{f(a) - f(a-h)}{h}$ is yet another representation of the slope of the tangent line to $y = f(x)$ at $x = a$. You can assume $h > 0$.

7. A particle moves along a straight line with equation of motion $s = f(t) = 2t^3 - t$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

8. Find the horizontal and vertical asymptotes of the function: $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

9. Consider the function $g(x)$ given below (c is a constant). Determine the value of c which makes $g(x)$ continuous on $(-\infty, \infty)$.

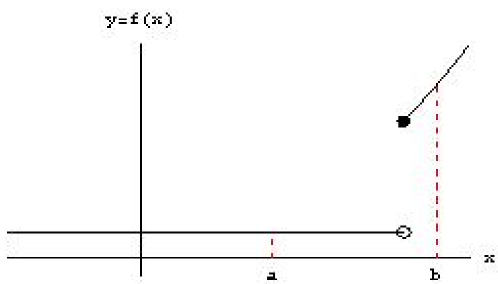
$$g(x) = \begin{cases} x^2 - c^2, & \text{if } x < 4 \\ cx + 20, & \text{if } x \geq 4 \end{cases}$$

10. Calculate the following using the limit laws $\displaystyle\lim_{x\to 2} \frac{\left(\frac{1}{x} - \frac{1}{2}\right)}{x-2}$.

11. Calculate the following using the limit laws $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+3x+1} - x\right)$.

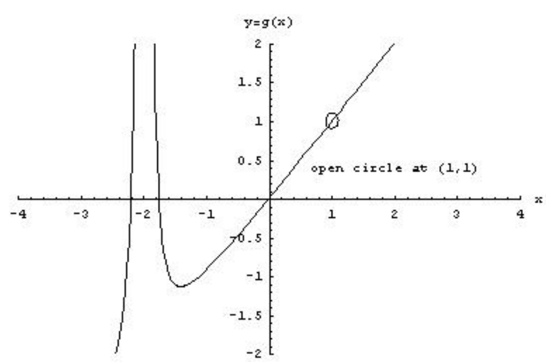
Solutions

Problem 1. Answers to this question may vary. The sketch of $f(x)$



shows a function which has a jump discontinuity in the interval $[a, b]$. Therefore, the Intermediate Value Theorem does not apply (since the function must be continuous).

Problem 2. Answers to this question may vary.



The sketch of $g(x)$ has the desired properties. Note that the slope of the tangent line to the curve at $x = 0$ is 1. This is what the condition $g'(0) = 1$ means.

Problem 3. If $f-g$ is continuous at a, we must have $\lim_{x\to a}(f-g)(x)=(f-g)(a)$. This is what we must show by our proof.

We are told that f and g are continuous at a, so we know that $\lim_{x\to a} f(x)=f(a)$ and $\lim_{x\to a} g(x)=g(a)$.

$$\begin{aligned}
\lim_{x\to a}(f-g)(x) &= \lim_{x\to a}\Big(f(x)-g(x)\Big) \quad \text{(algebra of functions)}\\
&= \Big[\lim_{x\to a} f(x)\Big]-\Big[\lim_{x\to a} g(x)\Big] \quad \text{(limit laws)}\\
&= [f(a)]-[g(a)] \quad \text{(since } f \text{ and } g \text{ are continuous)}\\
&= f(a)-g(a) \quad \text{(rewrite)}\\
&= (f-g)(a) \quad \text{(algebra of functions)}
\end{aligned}$$

So since $\lim_{x\to a}(f-g)(x)=(f-g)(a)$, the function $f-g$ is continuous.

Problem 4.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{h} \longrightarrow \frac{0}{0} \quad \text{indeterminant form; do some algebra}\\
&= \lim_{h\to 0}\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{h}\cdot\left(\frac{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}}{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}}\right)\\
&= \lim_{h\to 0}\frac{\big((x+h)^2-1\big)-\big(x^2-1\big)}{h(\sqrt{(x+h)^2-1}+\sqrt{x^2-1})}\\
&= \lim_{h\to 0}\frac{\cancel{x^2}+h^2+2xh-\cancel{1}-\cancel{x^2}+\cancel{1}}{h(\sqrt{(x+h)^2-1}+\sqrt{x^2-1})}\\
&= \lim_{h\to 0}\frac{\cancel{h}(h+2x)}{\cancel{h}(\sqrt{(x+h)^2-1}+\sqrt{x^2-1})}\\
&= \lim_{h\to 0}\frac{h+2x}{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}} \quad \text{now direct substitution will work}\\
&= \frac{0+2x}{\sqrt{(x+0)^2-1}+\sqrt{x^2-1}}\\
&= \frac{2x}{2\sqrt{x^2-1}}\\
&= \frac{x}{\sqrt{x^2-1}}
\end{aligned}$$

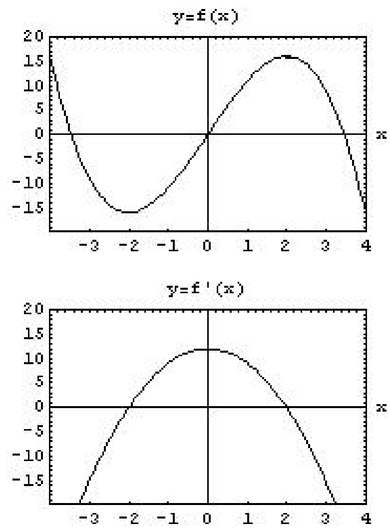
Problem 5. The major fact used in this solution is that the value of the derivative at $x=a$ is equal to the slope of the tangent line to the curve at $x=a$.

The curve $f(x)$ has a horizontal tangent at $x=\pm 2$, therefore the derivative will be zero there. That is, $f'(2)=f'(-2)=0$.

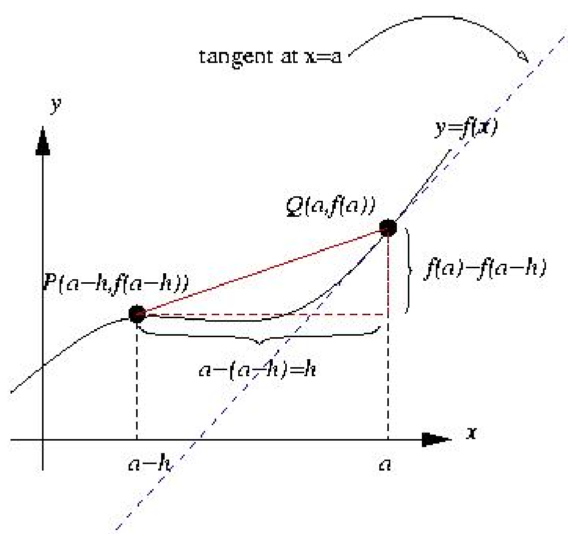
For $x\in(-4,-2)$ the function $f(x)$ is decreasing (negative slope); therefore the derivative will be negative in this region. That is, $f'(x)<0$.

For $x \in (-2, 2)$ the function $f(x)$ is increasing (positive slope); therefore the derivative will be positive in this region. That is, $f'(x) > 0$.

For $x \in (2, 4)$ the function $f(x)$ is decreasing (negative slope); therefore the derivative will be negative in this region. That is, $f'(x) < 0$.



Problem 6. We need a sketch that leads to $\lim_{h\to 0} \dfrac{f(a) - f(a-h)}{h}$. Since we can assume $h > 0$, that means we will have $a - h < a$ on our sketch.



The slope of the line through PQ is $\dfrac{f(a) - f(a-h)}{h}$.

If we take the limit as h approaches 0, the line through PQ approaches the tangent line at $x = a$, and therefore the tangent line to $y = f(x)$ at $x = a$ has the slope $m = \lim_{h\to 0} \dfrac{f(a) - f(a-h)}{h}$.

Problem 7. The velocity is equal to the derivative of the position.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
f(t) &= 2t^3 - t \\
f(a) &= 2a^3 - a \\
f(a+h) &= 2(a+h)^3 - (a+h) \\
&= 2(a^3+3a^2h+3ah^2+h^3) - a - h \\
&= 2a^3+6a^2h+6ah^2+2h^3-a-h \\
f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0}\frac{(2a^3+6a^2h+6ah^2+2h^3-a-h)-(2a^3-a)}{h} \quad \text{Direct substitution won't work} \\
&= \lim_{h\to 0}\frac{1}{h}[\cancel{2a^3}+6a^2h+6ah^2+2h^3-\cancel{a}-h-\cancel{2a^3}+\cancel{a}] \\
&= \lim_{h\to 0}\frac{1}{h}[6a^2h+6ah^2+2h^3-h] \\
&= \lim_{h\to 0}\frac{1}{\cancel{h}}[\cancel{h}(6a^2+6ah+2h^2-1)] \\
&= \lim_{h\to 0}(6a^2+6ah+2h^2-1) \quad \text{Direct substitution now works!} \\
&= 6a^2+6a(0)+2(0)^2-1 \\
&= 6a^2-1
\end{aligned}$$

The velocity when $t=a$ s is $v(a)=f'(a)=6a^2-1$. When $t=2$, the velocity is $6(2)^2-1=23$ m/s.

Problem 8. Horizontal Asymptotes:

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} &= \longrightarrow \frac{\infty}{\infty} \quad \text{direct substitution does not work; indeterminant form} \\
&= \lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{1/x}{1/x} \\
&= \lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{1/\sqrt{x^2}}{1/x} \quad \text{since } x=\sqrt{x^2} \text{ for } x>0 \\
&= \lim_{x\to\infty}\frac{\sqrt{2+1/x^2}}{3-5/x} \\
&= \frac{\sqrt{2+0}}{3-0} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}$$

So $y=\sqrt{2}/3$ is a horizontal asymptote.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5} &= \longrightarrow \frac{\infty}{-\infty} \quad \text{direct substitution does not work; indeterminant form} \\
&= \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{1/x}{1/x}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{1/(-\sqrt{x^2})}{1/x} \quad \text{since } x = -\sqrt{x^2} \text{ for } x < 0 \\
&= -\lim_{x\to-\infty} \frac{\sqrt{2+1/x^2}}{3-5/x} \\
&= -\frac{\sqrt{2+0}}{3-0} \\
&= -\frac{\sqrt{2}}{3}
\end{aligned}$$

So $y = -\sqrt{2}/3$ is a horizontal asymptote.

Vertical Asymptotes: The function might be infinite where the denominator is zero. So $x = 5/3$ is a possible vertical asymptote.

$\lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$, since $\sqrt{2x^2+1} > 0$ and finite at $x = 5/3$, and $3x-5 > 0$ for $x \geq 5/3$.

$\lim_{x\to\frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$, since $\sqrt{2x^2+1} > 0$ and finite at $x = 5/3$, and $3x-5 < 0$ for $x \leq 5/3$.

There is a vertical asymptote at $x = 5/3$.

Problem 9. If a function g is continuous at $x = a$, then $\lim_{x\to a^+} g(x) = \lim_{x\to a^-} g(x) = g(a)$. Our function $g(x)$ is piecewise defined. For $x < 4$, it is the polynomial $x^2 - c^2$, so it is continuous (polynomials are continuous). For $x > 4$ it is also a polynomial, so it will also be continuous in this region. The only point we don't know if the function $g(x)$ is continuous is at $x = 4$, not surprisingly the point where the definition changes.

We must choose c to make the function continuous at $x = 4$. We do this by by imposing that the following limits be equal: $\lim_{x\to4^+} g(x) = \lim_{x\to4^-} g(x)$.

Insert the proper definitions for $g(x)$: $\lim_{x\to4^+}(cx+20) = \lim_{x\to4^-}(x^2-c^2)$.

Evaluate by direct substitution: $c(4) + 20 = (4)^2 - c^2$.

A little algebraic rearranging gives us the following quadratic in c: $c^2 + 4c + 4 = 0$.

So if c satisfies this quadratic, then the left and right hand limits will be equal. The equality with $g(a)$ that is required for continuity follows automatically in this case. All that is left to do is solve the quadratic for c:

$$c(4) + 20 = (4)^2 + c^2 \quad \longrightarrow \quad (c+2)^2 = 0 \quad \longrightarrow \quad c = -2$$

So if $c = -2$, the function $g(x)$ will be continuous for $x \in \mathbb{R}$.

Problem 10.

$$\lim_{x\to2} \frac{\left(\frac{1}{x} - \frac{1}{2}\right)}{x-2} \quad \to \quad \frac{0}{0} \quad \text{direct substitution yields indeterminant quotient}$$

$$
\begin{aligned}
&= \lim_{x\to 2} \frac{\dfrac{2}{2x} - \dfrac{x}{2x}}{x-2} \\
&= \lim_{x\to 2} \frac{\left(\dfrac{2-x}{2x}\right)}{x-2} \\
&= -\lim_{x\to 2} \frac{1}{x-2}\left(\frac{x-2}{2x}\right) \\
&= -\lim_{x\to 2} \frac{1}{2x} \quad \text{direct substitution will now work} \\
&= -\frac{1}{2(2)} = -\frac{1}{4}
\end{aligned}
$$

Problem 11.

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right) &\to \infty-\infty \quad \text{direct substitution yields indeterminant difference} \\
&= \lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)\cdot\left(\frac{\sqrt{x^2+3x+1}+x}{\sqrt{x^2+3x+1}+x}\right) \quad \text{rationalize the numerator} \\
&= \lim_{x\to\infty}\frac{x^2+3x+1-x^2}{\sqrt{x^2+3x+1}+x} \\
&= \lim_{x\to\infty}\frac{3x+1}{\sqrt{x^2+3x+1}+x} \\
&\to \frac{\infty}{\infty} \quad \text{direct substitution yields indetermninate quotient} \\
&= \lim_{x\to\infty}\frac{3x+1}{\sqrt{x^2+3x+1}+x}\,\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \\
&= \lim_{x\to\infty}\frac{3+\frac{1}{x}}{\sqrt{x^2+3x+1}\frac{1}{x}+1}
\end{aligned}
$$

Since $x > 0$ (approaching infinity), we can say that $x = +\sqrt{x^2}$.

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right) &= \lim_{x\to\infty}\frac{3+\frac{1}{x}}{\sqrt{x^2+3x+1}\frac{1}{\sqrt{x^2}}+1} \\
&= \lim_{x\to\infty}\frac{3+\frac{1}{x}}{\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}+1}
\end{aligned}
$$

We can use the fact that $\lim_{x\to\infty}\dfrac{1}{x^r} = 0$ to get

$$
\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right) = \frac{3+0}{\sqrt{1+0+0}+1} = \frac{3}{2}.
$$