This is a set of practice test problems for Chapter 4. This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. I will post the solutions on my web site. The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words–remember, you can get part marks for talking about a problem!

There will be five problems on the test. Most will involve more than one part. You will have 100 minutes to complete the test. You may not use *Mathematica* or calculators on this test.

1. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = x^2 + \frac{2}{x}, \quad [\frac{1}{2}, 2]$$

2. Given

$$f(x) = xe^{-x}, \qquad f'(x) = (1-x)e^{-x} \qquad f''(x) = (x-2)e^{-x}$$

(a) find the intervals of increase or decrease
(b) find any local maximum or minimum values
(c) find the intervals of concavity and any inflection points
(d) find any vertical and horizontal asymptotes
(e) sketch the graph of $f(x)$

3. Use L'Hospital's Rule to find

$$\lim_{x\to-\infty} x^2 e^x$$

4. Use logarithms and L'Hospital's Rule to find

$$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$$

5. Use L'Hospital's Rule to show that

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

6. A farmer with 750 ft of fencing wants to enclose a rectangular area with one side of the rectangle against her barn. She then wants to divide this rectangle into four pens with fencing parallel to one side of the rectangle, where each pen is accessible from the barn. What is the largest possible total area of the four pens?

7. A rectangular poster is to have an area of 180 in^2 with one inch margins at the bottom and sides and a 2 inch margin at the top. What dimensions will give the largest printed area?

8. A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

9. Newton's Method approximates roots x_r of $f(x_r) = 0$ by iterating the equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Describe in detail (with diagrams and words) three ways that the method can fail.

10. A particle is moving with the given data. Find the position of the particle.

$$a(t) = 10 + \cos t + \sqrt{t}, s(0) = 0, s(\pi) = 1$$

11. Find $f(x)$ given that

$$f''(x) = 3e^x + 5\sin x, f(0) = 1, f'(\pi/2) = 2.$$

Solutions

Problem 1. $f(x)$ is continuous on the interval, and the interval is closed. We can find extrema by looking at $f(x)$ at the endpoints and at any critical numbers in the interval (aside: the name for this is the Closed Interval Method).

Critical numbers are when $f'(x) = 0$ or does not exist.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^2 + \frac{2}{x}\right] \\ &= 2x - \frac{2}{x^2} \end{aligned}$$

$f'(x)$ does not exist when $x = 0$, so $x = 0$ is a critical number. However, $x = 0$ is not in the interval $[1/2, 2]$, so we will not consider it.

Solve $f'(x) = 0$ for x:

$$\begin{aligned} f'(x) &= 0 \\ 2x - \frac{2}{x^2} &= 0 \\ \frac{2x^3 - 2}{x^2} &= 0 \\ 2x^3 - 2 &= 0 \quad \text{if } x \neq 0 \\ x^3 &= 1 \\ x &= \pm 1 \end{aligned}$$

The critical number $x = -1$ is outside the interval $[1/2, 2]$. Therefore, the only critical number we have inside the interval $[1/2, 2]$ is $x = 1$.

$$\begin{aligned} f(1/2) &= \left(\frac{1}{2}\right)^2 + \frac{2}{1/2} = \frac{1}{4} + 4 = 4.25 \\ f(2) &= (2)^2 + \frac{2}{2} = 4 + 1 = 5 \\ f(1) &= (1)^2 + \frac{2}{1} = 1 + 2 = 3 \end{aligned}$$

The absolute maximum is 5 at $x = 2$.
The absolute minimum is 3 at $x = 1$.

Problem 2.

- Intervals of Increasing/Decreasing:

Solve $f'(c) = (1 - c)e^{-c} = 0$. Since $e^{-x} \neq 0$, the only solution is $c = +1$. This is the only critical number for $f'(x)$ since $f'(x)$ exists for all x.

Write down a table showing where $f(x)$ is increasing and decreasing:

Interval	$f'(a)$ (a is in interval)	Sign of f'	f
$(-\infty, 1)$	$f'(0) = 1$	$+$	increasing
$(1, \infty)$	$f'(3) = -2e^{-3}$	$-$	decreasing

- Max/Min:

f goes from increasing to decreasing at $x = 1 \longrightarrow$ local max. $f(1) = e^{-1} = \frac{1}{e}$. Point: $(1, 1/e)$

- Intervals of Concave Up/Concave Down:

Solve $f''(c) = (c - 2)e^{-c} = 0$. Since $e^{-x} \neq 0$, the only solution is $c = +2$. This is the only critical number for $f''(x)$ since $f''(x)$ exists for all x.

Write down a table showing where $f(x)$ is concave up and down:

Interval	$f''(a)$ (a is in interval)	Sign of f''	f
$(-\infty, 2)$	$f''(0) = -2$	$-$	Concave Down
$(2, \infty)$	$f''(3) = e^{-3}$	$+$	Concave Up

• Points of Inflection:
The function f goes from concave down to concave up at $x = 2 \longrightarrow$ point of inflection. $f(2) = 2e^{-2} = 2/e^2$. Point: $(2, f(2)) = (2, 2/e^2)$

• Horizontal Asymptotes:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\left(xe^{-x}\right) \longrightarrow \infty\cdot 0 \quad \text{indeterminant product}\\
&= \lim_{x\to\infty}\left(\frac{x}{e^x}\right) \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminant quotient; use L'Hospital's Rule}\\
&= \lim_{x\to\infty}\left(\frac{\frac{d}{dx}[x]}{\frac{d}{dx}[e^x]}\right)\\
&= \lim_{x\to\infty}\left(\frac{1}{e^x}\right)\\
&= 0 \quad \text{since } \lim_{x\to\infty} e^x = \infty
\end{aligned}$$

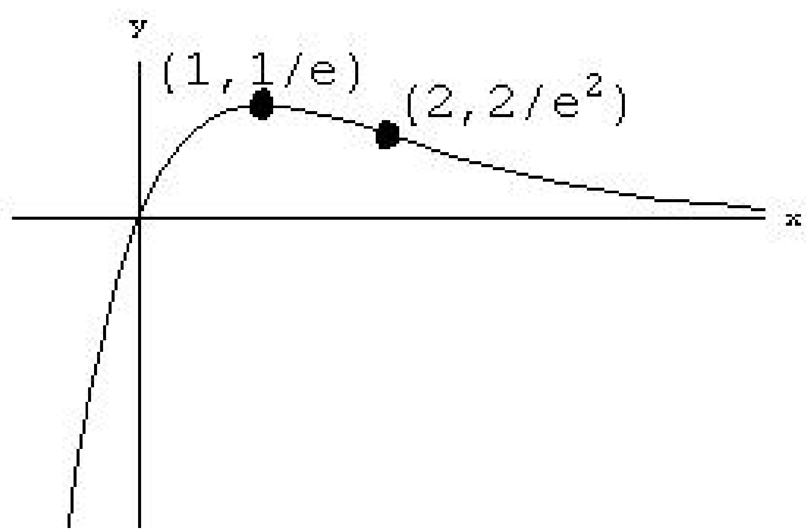
The line $y = 0$ is a horizontal asymptote.

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty}\left(xe^{-x}\right)\\
&\to (-\infty)(\infty) = -\infty
\end{aligned}$$

There is no horizontal asymptote as $x \longrightarrow -\infty$.
• Vertical Asymptotes:

$$\lim_{x\to a} f(x) = \pm\infty \longrightarrow x = a \text{ is a vertical asymptote}$$

This will not happen for our function. Our function has no vertical asymptotes.
• Sketch: Putting everything together from our detailed analysis, we get



Problem 3.

$$\begin{aligned}
\lim_{x\to-\infty} x^2e^x &\to \infty\cdot 0 \quad \text{indeterminant product} \\
&= \lim_{x\to-\infty}\frac{x^2}{e^{-x}} \to \frac{\infty}{\infty} \quad \text{indeterminant quotient; use L'Hospital's Rule} \\
&= \lim_{x\to-\infty}\frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} \\
&= \lim_{x\to-\infty}\frac{2x}{-e^{-x}} \\
&= -\lim_{x\to-\infty}\frac{2x}{e^{-x}} \to \frac{\infty}{\infty} \quad \text{indeterminant quotient; use L'Hospital's Rule} \\
&= -\lim_{x\to-\infty}\frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[e^{-x}]} \\
&= -\lim_{x\to-\infty}\frac{2}{-e^{-x}} \\
&= \lim_{x\to-\infty}\frac{2}{e^{-x}} \\
&= 0
\end{aligned}$$

Problem 4.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x &\to \left(\frac{\infty}{\infty}\right)^\infty \quad \text{? Use logarithms to get the power down} \\
y &= \left(\frac{x}{x+1}\right)^x \\
\ln y &= \ln\left[\left(\frac{x}{x+1}\right)^x\right] \\
\ln y &= x\ln\left(\frac{x}{x+1}\right) \\
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)
\end{aligned}$$

Aside:

$$\lim_{x\to\infty}\frac{x}{x+1} = \lim_{x\to\infty}\frac{1}{1+1/x} = \frac{1}{1+0} = 1$$

$$\begin{aligned}
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) \\
&\to \infty\ln(1) = \infty\cdot 0 \quad \text{indeterminant product} \\
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} \to \frac{0}{0} \quad \text{indeterminant quotient; use L'Hospital's Rule} \\
&= \lim_{x\to\infty}\frac{\dfrac{d}{dx}\left[\ln\left(\dfrac{x}{x+1}\right)\right]}{\dfrac{d}{dx}\left[\dfrac{1}{x}\right]}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{\dfrac{1}{x/(x+1)}\dfrac{d}{dx}\left[\dfrac{x}{x+1}\right]}{\left(-\dfrac{1}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{x+1}{x}(-x^2)\frac{(x+1)\dfrac{d}{dx}[x] - x\dfrac{d}{dx}[x+1]}{(x+1)^2} \\
&= \lim_{x\to\infty} \frac{-x}{x+1} \\
&= \lim_{x\to\infty} \frac{-1}{1+\frac{1}{x}} = \frac{-1}{1+0} = -1
\end{aligned}$$

So $\lim_{x\to\infty} \ln y = -1$.
However, we wanted

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x &= \lim_{x\to\infty} y \\
&= \lim_{x\to\infty} e^{\ln y} \\
&= e^{\lim_{x\to\infty} \ln y} \\
&= e^{-1} \\
&= \frac{1}{e}
\end{aligned}$$

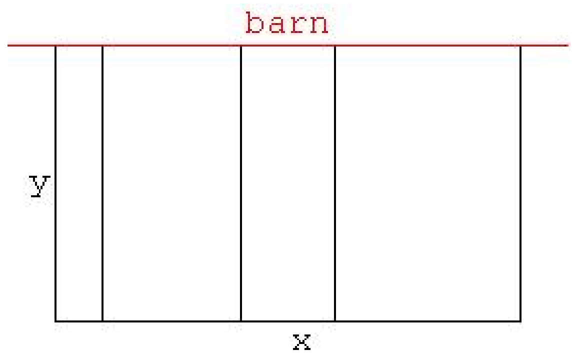
Problem 5.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} &\to \frac{0}{0} \quad \text{indetereminant quotient; use L'Hospital's Rule} \\
&= \lim_{h\to 0}\frac{\dfrac{d}{dh}[f(x+h)-2f(x)+f(x-h)]}{\dfrac{d}{dh}[h^2]} \\
&= \lim_{h\to 0}\frac{\dfrac{d}{dh}[f(x+h)]+\dfrac{d}{dh}[f(x-h)]}{2h} \\
&= \lim_{h\to 0}\frac{f'(x+h)\dfrac{d}{dh}[x+h]+f'(x-h)\dfrac{d}{dh}[x-h]}{2h} \quad \text{chain rule} \\
&= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \to \frac{0}{0} \quad \text{indeterminant quotient; use L'HR} \\
&= \lim_{h\to 0}\frac{\dfrac{d}{dh}[f'(x+h)-f'(x-h)]}{\dfrac{d}{dh}[2h]} \\
&= \lim_{h\to 0}\frac{f''(x+h)\dfrac{d}{dh}[x+h]-f''(x-h)\dfrac{d}{dh}[x-h]}{2} \\
&= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2} \\
&= \frac{f''(x+0)+f''(x-0)}{2} \\
&= \frac{2f''(x)}{2} = f''(x)
\end{aligned}$$

Problem 6. This is an optimization problem where we want to maximize the total area enclosed.

The perimeter of the fencing must be 750 ft. This is the constraint we will use to eliminate any extra variables in the expression for the total area, since we only know how to optimize a function of one variable.

Diagram:



Notice that the pens do not have to be the same size!

The total area is $A = xy$.
The total perimeter is $P = 5y + x = 750$ (the barn side does not require fencing, so we get x instead of $2x$).

Let's write $x = 750 - 5y$ and the total area can be expressed as a function of one variable:

$$A(y) = (750 - 5y)y = 750y - 5y^2.$$

Since this function is a parabola opening down, when we find the extrema it will be a maximum. Also, it must be an absolute maximum due to the shape of a parabola.

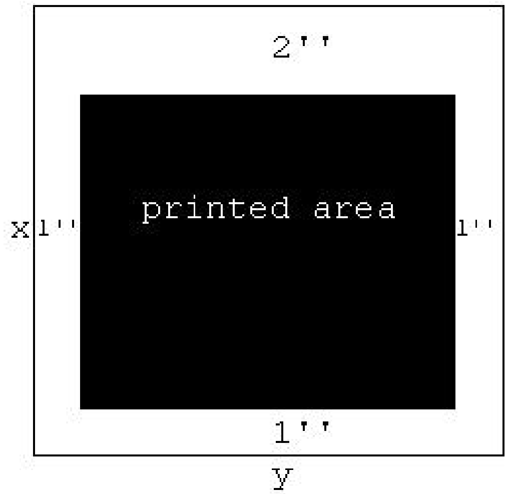
To find an extrema, solve $A'(y) = 0$ for y.

$$\begin{aligned} A(y) &= 750y - 5y^2 \\ A'(y) &= \frac{d}{dy}[750y - 5y^2] \\ &= 750 - 10y \\ 0 &= 750 - 10y \\ y &= 75 \end{aligned}$$

Therefore, the maximum are will be when $y = 75$ ft, or $A(75) = 28125$ ft^2.

Problem 7. Here we are trying to maximize the printed area.

Diagram:



Area of printed region $= A = (x-3)(y-2)$.
Area of poster is $= xy = 180$ in^2.

Let's write $y = 180/x$ and the total area can be expressed as a function of one variable:

$$A(x) = (x-3)\left(\frac{180}{x} - 2\right) = 180 - \frac{540}{x} - 2x + 6 = 186 - 2x - \frac{540}{x}.$$

To find an extrema, solve $A'(x) = 0$ for x.

$$\begin{aligned} A'(x) &= -2 + \frac{540}{x^2} \\ 0 &= \frac{-2x^2 + 540}{x^2} \\ 0 &= -2x^2 + 540 \\ x &= \sqrt{270} \text{ in.} \end{aligned}$$

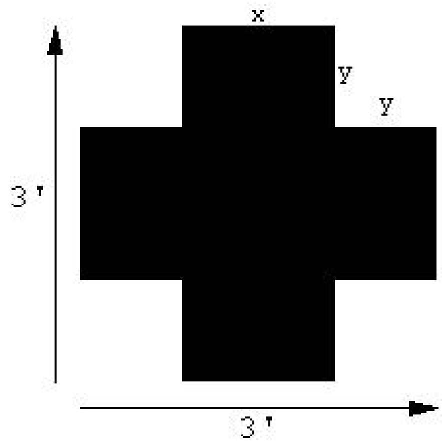
We must answer the question of whether this value of x produces a maximum or a minimum. Let's show that at $x = \sqrt{270}$ $A(x)$ is concave down, and so we will have a maximum.

$$\begin{aligned} A'(x) &= -2 + \frac{540}{x^2} \\ A''(x) &= -\frac{1080}{x^3} \\ A''(\sqrt{270}) &= -\frac{1080}{(270)^{3/2}} < 0 \end{aligned}$$

Since the second derivative is less than zero at $x = \sqrt{270}$, we know that the function is concave down at that point. Therefore we have found a maximum. The geometry of the situation (area of a rectangle) tells us that we will have a single maximum, therefore we have an absolute maximum if the poster has dimensions $x = \sqrt{270}$ in, $y = 180/\sqrt{270}$ in.

Problem 8. If we construct a box as the problem suggests, all the squares which are cut out must have the same shape. Let's say these squares which are cut out are $y \times y$ ft^2.

Diagram:



The other dimension on the flap we will label as x. This leads to the relation $x + 2y = 3$.
When we fold the sides up, the volume of the box that is created will be $V = x^2 y$.
Let's write $x = 3 - 2y$ and the total volume can be expressed as a function of one variable:

$$V(y) = (3 - 2y)^2 y = (9 - 12y + 4y^2)y = 9y + 4y^3 - 12y^2.$$

To find an extrema, solve $V'(y) = 0$ for y.

$$\begin{aligned} V(y) &= 9y + 4y^3 - 12y^2 \\ V'(y) &= 9 + 12y^2 - 24y \\ V'(y) &= 9 + 12y^2 - 24y \\ 0 &= 12y^2 - 24y + 9 \end{aligned}$$

Using the quadratic formula, we can solve for y:

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{24 \pm \sqrt{(24)^2 - 4(12)(9)}}{2(12)} \\ &= \frac{24 \pm 12}{24} \\ &= \frac{36}{24} \text{ or } \frac{12}{24} \\ &= \frac{3}{2} \text{ or } \frac{1}{2} \end{aligned}$$

Which one is the maximum? Are they both maximums? Let's answer this question by looking at the concavity.

$$\begin{aligned} V''(y) &= 24y - 24 \\ V''(1/2) &= 24(1/2) - 24 = -12 < 0 \quad \text{concave down, so } y = 1/2 \text{ produces a max} \\ V''(3/2) &= 24(3/2) - 24 = +12 > 0 \quad \text{concave up, so } y = 3/2 \text{ produces a min} \end{aligned}$$

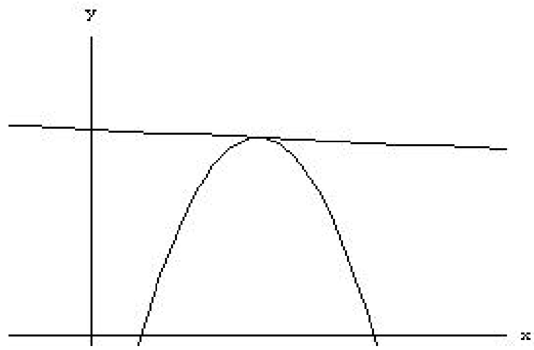
Therefore, the maximum volume of the box is $V(1/2) = 2 \text{ in}^3$.

Another way to argue that $y = 1/2$ produces the max is that if $y = 3/2$ then $x = 3 - 2y = 0$, and the box becomes unphysical (zero volume). Also, if $y = 0$, the box is unphysical (zero volume).

Therefore, function we wish to maximize if $V(y) = 9y + 4y^3 - 12y^2, y \in [0, 3/2]$, which is continuous on its domain.
$V(0) = 0$.
$V(1/2) = 2$.
$V(3/2) = 0$.
So the maximum is at $y = 1/2$ by the Closed Interval Theorem.

Problem 9. Here are three problems with Newton's Method:

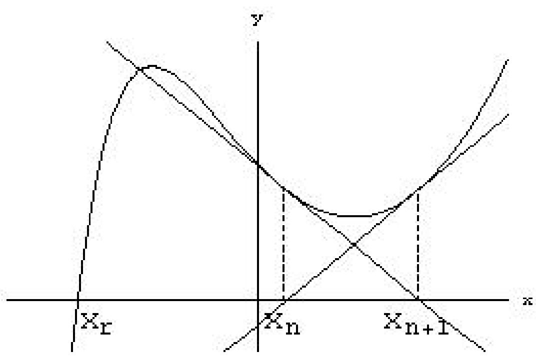
i) If $f'(x_n) \sim 0$, you can get a situation like the following.



The next approximation could be extremely far away from the root!

ii) If you have two roots which are close together, you cannot tell which root Newton's Method has found.

iii) Newton's method can get "stuck" in the following manner (or in more complicated manners), and just oscillate forever without finding the root x_r:



Problem 10. This is an antidifferentiation problem

$$
\begin{aligned}
a(t) &= 10 + \cos t + t^{1/2} \\
v(t) &= 10t + \sin t + \frac{t^{3/2}}{3/2} + c_1 \\
v(t) &= 10t + \sin t + \frac{2}{3}t^{3/2} + c_1 \\
s(t) &= 5t^2 - \cos t + \frac{2}{3}\frac{t^{5/2}}{5/2} + c_1 t + c_2 \\
s(t) &= 5t^2 - \cos t + \frac{4}{15}t^{5/2} + c_1 t + c_2
\end{aligned}
$$

Now we use the conditions to determine the constants c_1 and c_2:

$$\begin{aligned} s(0) &= 5(0)^2 - \cos 0 + \frac{4}{15}(0)^{5/2} + c_1(0) + c_2 \\ 0 &= -1 + c_2 \\ c_2 &= 1 \end{aligned}$$

$$\begin{aligned} s(\pi) &= 5(\pi)^2 - \cos \pi + \frac{4}{15}(\pi)^{5/2} + c_1(\pi) + 1 \\ 0 &= 5\pi^2 + 1 + \frac{4}{15}\pi^{5/2} + c_1\pi + 1 \\ c_1 &= -\frac{1}{\pi} - 5\pi - \frac{4}{15}\pi^{3/2} \end{aligned}$$

Therefore,

$$s(t) = 5t^2 - \cos t + \frac{4}{15}t^{5/2} - t(-\frac{1}{\pi} - 5\pi - \frac{4}{15}\pi^{3/2}) + 1$$

Problem 11. This is an antiderivative problem.

$$\begin{aligned} f''(x) &= 3e^x + 5\sin x \\ f'(x) &= 3e^x - 5\cos x + c_1 \\ f(x) &= 3e^x - 5\sin x + c_1 x + c_2 \end{aligned}$$

Now we use the conditions to determine the constants c_1 and c_2:

$$\begin{aligned} f(0) &= 3e^0 - 5\sin 0 + c_1(0) + c_2 \\ 1 &= 3 + c_2 \\ c_2 &= -2 \end{aligned}$$

$$\begin{aligned} f'(\pi/2) &= 3e^{\pi/2} - 5\cos \pi/2 + c_1 \\ 2 &= 3e^{\pi/2} + c_1 \\ c_1 &= 2 - 3e^{\pi/2} \end{aligned}$$

Therefore,

$$f(x) = 3e^x - 5\sin x + (2 - 3e^{\pi/2})x - 2$$