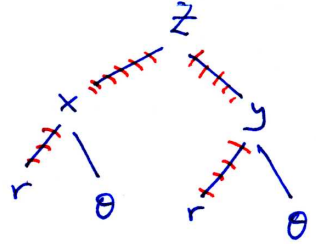9. (15pts.) Let $z = f(x, y)$ be a function that has continuous second order partial derivatives. Let $x = r^3\theta$, $y = r\theta^2$.

(a) (7pts.) Find the following first order partial derivative in terms of r, θ, and the first order partials of z with respect to x and y ONLY.

(**Hint:** Sketch a tree diagram if it helps.)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x} \cdot 3r^2\theta + \frac{\partial z}{\partial y} \theta^2$$

i.e. $\frac{\partial z}{\partial x} 3r^2\theta + \frac{\partial z}{\partial y} \cdot \theta^2$ or $\boxed{3r^2\theta \frac{\partial z}{\partial x} + \theta^2 \frac{\partial z}{\partial y}}$

(b) (8pts.) Given that the first order partial derivative of z with respect to θ is

$$\frac{\partial z}{\partial \theta} = r^3 \frac{\partial z}{\partial x} + 2r\theta \frac{\partial z}{\partial y}$$

find the following second order partial derivative in terms of r, θ, and the first and second order partials of z with respect to x and y ONLY.

(**Hint:** Sketch tree diagrams.)

$$\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial \theta}\right] = \frac{\partial}{\partial r}\left[r^3 \frac{\partial z}{\partial x} + 2r\theta \frac{\partial z}{\partial y}\right]$$

$$= 3r^2 \frac{\partial z}{\partial x} + r^3 \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\right] + 2\theta \frac{\partial z}{\partial y} + 2r\theta \cdot \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\right]$$

$$= 3r^2 \frac{\partial z}{\partial x} + r^3\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial y}{\partial r}\right] + 2\theta \frac{\partial z}{\partial y}$$

$$+ 2r\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \cdot \frac{\partial y}{\partial r}\right]$$

$$= 3r^2 \frac{\partial z}{\partial x} + 2\theta \frac{\partial z}{\partial y} + r^3\left[\frac{\partial^2 z}{\partial x^2} 3r^2\theta + \frac{\partial^2 z}{\partial y \partial x} \theta^2\right]$$

$$+ 2r\theta\left[\frac{\partial^2 z}{\partial x \partial y} \cdot 3r^2\theta + \frac{\partial^2 z}{\partial y^2} \cdot \theta^2\right]$$

$$= 3r^2 \frac{\partial z}{\partial x} + 2\theta \frac{\partial z}{\partial y} + 3r^5\theta \frac{\partial^2 z}{\partial x^2} + r^3\theta^2 \frac{\partial^2 z}{\partial y \partial x} + 6r^3\theta^2 \frac{\partial^2 z}{\partial x \partial y} + 2r\theta^3 \frac{\partial^2 z}{\partial y^2}$$