CSci 1302 Example of a proof in predicate logic

The example uses the following predicates on integer numbers:

1. $next(x, y)$ is true if $y = x + 1$, false otherwise.

2. $greater(x, y)$ is true if $x > y$, false otherwise.

Below are two proofs of the same mathematical argument: if every number has the next number and the next number of x is greater than x, then there is no greatest number. Note that we need to add to our list of assumptions the property that if x is greater than y, then y is not greater than x. Without this assumption the proof does not go through.

There are two versions of the proof. The first one uses the theorems of predicate logic $\neg\exists x.p(x) <\equiv> \forall x.\neg p(x)$ and $\neg\forall x.p(x) <\equiv> \exists x.\neg p(x)$. The other one is a proof by contradiction.

1.	$\forall x.\exists y.next(x, y)$	*Assumption*
2.	$\forall x.\forall y.next(x, y) \Rightarrow greater(y, x)$	*Assumption*
3.	$\forall x.\forall y.greater(x, y) \Rightarrow \neg greater(y, x)$	*Assumption*
	$\neg\exists x\forall y.greater(x, y)$	
4.	$\exists y.next(x_\forall, y(x_\forall))$	$1, \forall\_E(gen), y$ depends on $x_\forall$
5.	$next(x_\forall, y_\exists(x_\forall))$	$4, \exists\_E$
6.	$next(x_\forall, y_\exists(x_\forall)) \Rightarrow greater(y_\exists(x_\forall), x_\forall)$	$2, \forall\_E(unknown)$ twice ($x_\forall$ appeared before, so it's unknown, not genuine)
7.	$greater(y_\exists(x_\forall), x_\forall)$	$5, 6,$ Modus Ponens
8.	$greater(y_\exists(x_\forall), x_\forall) \Rightarrow \neg greater(x_\forall, y_\exists(x_\forall))$	$3, \forall\_E(unknown)$ twice
9.	$\neg greater(x_\forall, y_\exists(x_\forall))$	$7, 8,$ Modus Ponens
10.	$\exists y.\neg greater(x_\forall, y)$	$9, \exists\_I$
11.	$\forall x.\exists y.greater(x, y)$	$10, \forall\_I$
12.	$\forall x.\neg\forall y.greater(x, y)$	$11,$ Thm. $\neg\forall x.p(x) <\equiv> \exists x.\neg p(x)$
13.	$\neg\exists x.\forall y.greater(x, y)$	$12,$ Thm. $\neg\exists x.p(x) <\equiv> \forall x.\neg p(x)$

Note that step 10 introduces $\exists$ quantifier for a variable $y_\exists$ which depends on $x_\forall$. This is OK. However, introducing the $\forall$ quantifier would not have been allowed. See example on p. 123 for the explanation of these rules.

The same example can be done using a proof by contradiction: assume that there is the greatest number, and find a contradiction.

1.	$\forall x.\exists y.next(x,y)$	*Assumption*
2.	$\forall x.\forall y.next(x,y) \Rightarrow greater(y,x)$	*Assumption*
3.	$\forall x.\forall y.greater(x,y) \Rightarrow \neg greater(y,x)$	*Assumption*
	$\neg\exists x \forall y.greater(x,y)$	
4.	$\exists x.\forall y.greater(x,y)$	*Assumption*(there is the largest number)
5.	$\forall y.greater(x_\exists, y)$	$4, \exists\_E$
6.	$\exists y.next(x_\exists, y)$	$1, \forall\_E(unknown)$
7.	$next(x_\exists, y_\exists)$	$6, \exists\_E$
8.	$next(x_\exists, y_\exists) \Rightarrow greater(y_\exists, x_\exists)$	$7, \forall\_E(unknown)$ twice
9.	$greater(y_\exists, x_\exists)$	$7, 8,$ Modus Ponens
10.	$greater(y_\exists, x_\exists) \Rightarrow \neg greater(x_\exists, y_\exists)$	$3, \forall\_E(unknown)$ twice
11.	$\neg greater(x_\exists, y_\exists)$	$9, 10,$ Modus Ponens
12.	$greater(x_\exists, y_\exists)$	$5, \forall\_E(unknown)$
13.	$\neg\exists x.\forall y.greater(x,y)$	$4-12,$ Contradiction lines 11, 12