CSci 1302 Assignment 5

Due Wedn., February 27th in class

For these problems assume the universal set is the set $\mathbb{Z}$ of all integers: positive, negative, and zero. $x < y$, $x \leq y$, etc. are binary predicates with the obvious meaning. We also use predicates $isOdd(x)$, $isEven(x)$, and $isDivisibleBy(x, y)$, which means that x is divisible by y.

Problem 1 (10 points). Translate the following formulas to English, indicate whether each one is true or false, and briefly justify your answer.

1. $\tilde{}\forall x.(x^2 > 0) \vee (x = 0)$
2. $\exists x.x^2 \leq x$
3. $\forall x.isOdd(x) \rightarrow isEven(x)$
4. $\exists x.(x \leq 2) \rightarrow (isOdd(x) \wedge isEven(x))$
5. $\exists x.(x \leq 2) \leftrightarrow isOdd(x)$
6. $\exists x.\exists y.x > 2 \wedge x + y < 0$
7. $\forall x.x \leq x$
8. $\forall x.\exists y.isDivisible(x, y)$
9. $\exists x.\forall y.isDivisible(y, x)$
10. $\forall x.\exists y.isOdd(x) \rightarrow isOdd(y)$

Problem 2 (10 points). Write the following sentences as quantified formulas. Note that some of these formulas need more than one quantifier.

1. Every number is divisible by 1.
2. Some numbers are divisible by 3.
3. Not all numbers are divisible by 3.
4. No odd number is divisible by 2.
5. No number is greater than itself.
6. Squares of odd numbers are odd.
7. No squares of even numbers are prime.
8. Every number is divisible by some number.

9. Some numbers are squares of some other numbers (don't use the predicate $isSquare(x)$).
10. No matter what pair of numbers you take, you can find a number that they both are divisible by.

Problem 3 (4 points). Which of the following formulas are equivalent to $\tilde{}\forall x.\exists y.p(x,y)$? Please explain your reasoning for each formula below.

1. $\forall x.\exists y.\tilde{}p(x,y)$
2. $\tilde{}\exists y.\forall x.p(x,y)$
3. $\exists x.\tilde{}\exists y.p(x,y)$
4. $\exists x.\forall y.\tilde{}p(x,y)$

Problem 4 (16 points: 1,2 are 2 points each, the rest are 3 points each). Assume the following:

1. A chess team A consists of Adam, Alice, and Ann. A(x) means that the person x is on the team A.
2. A chess team B consists of Bob and Beth. B(x) means that the person x is on the team B.
3. The universal set for the problem is the set of all five chess players.
4. The relation wonAgainst(x,y) means that x has won against y at least once. Some people never played against each other, so no comparison is given for such pairs. The following are true statements:
   (a) Adam has won against Ann and Alice.
   (b) Alice has won against Ann.
   (c) Beth has won against Adam and Bob.
   (d) Bob has won against Alice.
   (e) Ann has won against Alice.

   Note that since no person won against themselves, wonAgainst(x,x) is false for any x in the domain.

Based on the information above, are the following true or false statements? Prove your answers by showing all instances (single elements or pairs) necessary to prove or disprove the statement. When considering all possible pairs, it may be convenient to organize your answers as a table. You may show only a part of the table if it is sufficient to prove or disprove the statement.

1. $\exists x.A(x) \rightarrow wonAgainst(x, Beth)$
2. $\forall x.wonAgainst(Adam, x) \vee wonAgainst(Beth, x)$
3. $\forall x.\exists y.A(x) \rightarrow wonAgainst(y, x)$
4. $\exists x.\forall y.wonAgainst(x, y) \vee wonAgainst(y, x)$
5. $\exists x.\exists y.wonAgainst(x, y) \wedge wonAgainst(Adam, x)$
6. $\forall x.\forall y.A(x) \vee B(y)$

Problem 5 (Extra credit, 3 points) Use the system in Problem 4 to give an example of a predicate $p(x, y)$ such that $\forall x.\exists y.p(x, y)$ is true, but $\exists y.\forall x.p(x, y)$ is not. You may combine the given predicates in any way you want, but do not add any new predicates.