CSci 1302 Example of a proof in predicate logic

Below are two proofs of the same mathematical argument: if every number has the next number and the next number of x is greater than x, then there is no greatest number. Note that we need to add to our list of assumptions the property that if x is greater than y, then y is not greater than x. Without this assumption the proof does not go through.

There are two versions of the proof. The first one uses the theorems of predicate logic $\tilde{}\exists x.p(x) \equiv \forall x.\tilde{}p(x)$ and $\tilde{}\forall x.p(x) \equiv \exists x.\tilde{}p(x)$. The other one is a proof by contradiction.

1.	$\forall x.\exists y.y = x + 1$	
2.	$\forall x.\forall y.y = x + 1 \rightarrow (y > x)$	
3.	$\forall x.\forall y.(x > y) \rightarrow \tilde{}(y > x)$	
	$\therefore \tilde{}\exists x \forall y.x > y$	
4.	$\exists y.y(x_\forall) = x_\forall + 1$	$1, \forall\_E(gen), y$ depends on $x_\forall$
5.	$y(x_\forall) = x_\forall + 1$	$4, \exists\_E$
6.	$y(x_\forall) = x_\forall + 1 \rightarrow (y_\exists(x_\forall) > x_\forall)$	$2, \forall\_E(gen, unknown)$ ($x_\forall$ appeared before)
7.	$y_\exists(x_\forall) > x_\forall$	$5, 6,$ Modus Ponens
8.	$(y_\exists(x_\forall) > x_\forall) \rightarrow \tilde{}(x_\forall > y_\exists(x_\forall))$	$3, \forall\_E(unknown)$ twice
9.	$\tilde{}(x_\forall > y_\exists(x_\forall))$	$7, 8,$ Modus Ponens
10.	$\exists y.\tilde{}(x_\forall > y)$	$9, \exists\_I$
11.	$\forall x.\exists y.\tilde{}(x > y)$	$10, \forall\_I$
12.	$\forall x.\tilde{}\forall y.x > y$	$11, \tilde{}\forall x.p(x) \equiv \exists x.\tilde{}p(x)$
13.	$\tilde{}\exists x.\forall y.x > y$	$12, \tilde{}\exists x.p(x) \equiv \forall x.\tilde{}p(x)$

Note that step 10 introduces $\exists$ quantifier for a variable $y_\exists$ which depends on $x_\forall$. This is OK. However, introducing the $\forall$ quantifier would not have been allowed.

The same example can be done using a proof by contradiction: assume that there is the greatest number, and find a contradiction.

1.	$\forall x.\exists y.y = x + 1$	*Assumption*
2.	$\forall x.\forall y.y = x + 1 \rightarrow y > x$	*Assumption*
3.	$\forall x.\forall y.(x > y) \rightarrow \sim(y < x)$	*Assumption*
	$\sim\exists x\forall y.x > y$	
4.	$\exists x.\forall y.x > y$	*Assumption*(there is the largest number)
5.	$\forall y.x_\exists > y$	$4, \exists\_E$
6.	$\exists y.y = x_\exists + 1$	$1, \forall\_E(unknown)$
7.	$y_\exists = x_\exists + 1$	$6, \exists\_E$
8.	$y_\exists = x_\exists + 1 \rightarrow (y_\exists > x_\exists)$	$7, \forall\_E(unknown)$ twice
9.	$y_\exists > x_\exists$	$7, 8,$ Modus Ponens
10.	$(y_\exists > x_\exists) \rightarrow \sim(x_\exists > y_\exists)$	$3, \forall\_E(unknown)$ twice
11.	$\sim(x_\exists > y_\exists)$	$9, 10,$ Modus Ponens
12.	$x_\exists > y_\exists$	$5, \forall\_E(unknown)$
13.	$\sim\exists x.\forall y.x > y$	$4 - 12,$ Contradiction lines 11, 12