## 1 Quadratic Functions

In this unit we will learn many of the algebraic techniques used to work with the quadratic function

$$
f(x)=a x^{2}+b x+c .
$$

### 1.1 Completing the Square

The algebra process called completing the square is the way that we can isolate the variable $x$ in a general quadratic. You will see this process again in precalculus and calculus, and it is definitely necessary to solve certain kinds of problems.
I want you to see it now and be able to work with it for simple cases.

## Completing the Square:

1. Factor so that there is just a 1 in front of the $x^{2}$ term.
2. Identify the coefficient of the $x$ term.
3. Take half of this coefficient and square, then add and subtract so you don't change the equation.
4. Factor the perfect square you have created.
5. Simplify.

$$
\begin{aligned}
a x^{2}+b x+c & =a\left(x^{2}+\frac{b}{a} x\right)+c & & \text { (Step 1. factor) } \\
& =a\left(x^{2}+\frac{b}{a} x\right)+c & & \text { (Step 2. coefficient of } x \text { is } \frac{b}{a} \text { ) } \\
& =a\left(x^{2}+\frac{b}{a} x+\left(\frac{\boldsymbol{b}}{\mathbf{2 a}}\right)^{2}-\left(\frac{\boldsymbol{b}}{\mathbf{2 a}}\right)^{2}\right)+c & & \text { (Step 3. blue terms are adding zero) } \\
& =a\left(x^{2}+\frac{b}{a} x+\left(\frac{b}{2 a}\right)^{2}-\left(\frac{b}{2 a}\right)^{2}\right)+c & & \text { (the red terms are a perfect square) } \\
& =a\left(\left[x+\frac{b}{2 a}\right]^{2}-\left(\frac{b}{2 a}\right)^{2}\right)+c & & \text { (Step 4. factor perfect square) } \\
& =a\left[x+\frac{b}{2 a}\right]^{2}-a\left(\frac{b}{2 a}\right)^{2}+c & & \text { (Step 5. distribute the } a \text { ) } \\
& =a\left[x+\frac{b}{2 a}\right]^{2}-\frac{b^{2}}{4 a}+c & & \text { (simplify) } \\
& =a\left[x+\frac{b}{2 a}\right]^{2}-\frac{b^{2}-4 a c}{4 a} & & \text { (common denominator) }
\end{aligned}
$$

The process of completing the square shows that

$$
f(x)=a x^{2}+b x+c=a\left[x+\frac{b}{2 a}\right]^{2}-\frac{b^{2}-4 a c}{4 a}
$$

### 1.2 Solving Quadratic Equations: The Quadratic Formula

To solve simple quadratic equation of the form $x^{2}=$ constant, we can use the square root property.

## Square root property: <br> Solution to $x^{2}=a$ is $x= \pm \sqrt{a}$.

The square root property makes sense if you consider factoring $x^{2}=a$ :

$$
\begin{aligned}
x^{2}-\boldsymbol{a} & =\boldsymbol{\not}-\boldsymbol{a} & & \text { (addition principle) } \\
x^{2}-a & =0 & & \\
x^{2}-(\sqrt{a})^{2} & =0 & & \text { (properties of radicals) } \\
(x-\sqrt{a})(x+\sqrt{a}) & =0 & & \text { (factor as a difference of squares) } \\
x-\sqrt{a}=0 \text { or } x & +\sqrt{a}=0 & & \text { (zero factor property) } \\
x=\sqrt{a} \text { or } x & =-\sqrt{a} & &
\end{aligned}
$$

We can now derive the quadratic formula, which is used to solve more general quadratic equations of the form $a x^{2}+b x+c=0$.

$$
\begin{aligned}
& a x^{2}+b x+c=0 \\
& a\left[x+\frac{b}{2 a}\right]^{2}-\frac{b^{2}-4 a c}{4 a}=0 \\
& a\left[x+\frac{b}{2 a}\right]^{2}-\frac{b^{2}-4 a c}{4 a}+\frac{b^{2}-4 a c}{4 a}=+\frac{b^{2}-4 a c}{4 a} \quad \quad \text { (addition principle) } \\
& \boldsymbol{\alpha}\left[x+\frac{b}{2 a}\right]^{2} \times \frac{\mathbf{1}}{\boldsymbol{\alpha}}=\frac{b^{2}-4 a c}{4 a} \times \frac{\mathbf{1}}{\boldsymbol{a}} \quad \text { (division principle) } \\
& {\left[x+\frac{b}{2 a}\right]^{2}=\frac{b^{2}-4 a c}{4 a^{2}}} \\
& x+\frac{b}{2 a}= \pm \sqrt{\frac{b^{2}-4 a c}{4 a^{2}}} \quad \text { (square root principle) } \\
& x+\frac{b}{2 a}= \pm \frac{\sqrt{b^{2}-4 a c}}{2 a} \quad \text { (properties of radicals } \\
& x+\frac{b /}{2 a}-\frac{b}{2 a}=-\frac{\boldsymbol{b}}{2 \boldsymbol{a}} \pm \frac{\sqrt{b^{2}-4 a c}}{2 a} \quad \text { (addition principle) } \\
& x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \quad \text { (common denominator) }
\end{aligned}
$$

## Quadratic formula:

Solution to $a x^{2}+b x+c=0$ is $x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$.

There are three possibilities for the solution, based on the sign of the quantity $b^{2}-4 a c$.

For the equation $a x^{2}+b x+c=0$, the discriminant is defined as $b^{2}-4 a c$.
The equation $a x^{2}+b x+c=0$ has

1. Two real-valued solutions if $b^{2}-4 a c>0$.
2. One real-valued solution if $b^{2}-4 a c=0$.
3. Two complex-valued solutions if $b^{2}-4 a c<0$. The two complex-valued solutions will be complex conjugates.

EXAMPLE Use completing the square to solve $2 x^{2}+4 x+1=0$.

$$
\begin{array}{rlrl}
\frac{2 x^{2}+6 x+1}{2} & =\frac{0}{2} & & \text { (Step 1. factor, or divide to ensure } x^{2} \text { has coefficient of 1) } \\
x^{2}+3 x+\frac{1}{2} & =0 & & \\
x^{2}-3 x+\frac{1}{2} & =0 & & \text { (Step 2. coefficient of } x \text { is 3) } \\
x^{2}+3 x+\left(\frac{\mathbf{3}}{2}\right)^{2}-\left(\frac{\mathbf{3}}{\mathbf{2}}\right)^{2}+\frac{1}{2} & =0 & & \text { (Step 3. blue terms are adding zero) } \\
x^{2}+3 x+\left(\frac{\mathbf{3}}{\mathbf{2}}\right)^{2}-\left(\frac{3}{2}\right)^{2}+\frac{1}{2} & =0 & & \text { (Step 4. red terms are perfect square) } \\
\left(x+\frac{3}{2}\right)^{2}-\frac{9}{4}+\frac{2}{4} & =0 \\
\left(x+\frac{3}{2}\right)^{2}-\frac{7}{4}+\frac{7}{4} & =0+\frac{7}{4} & & \text { (Step 4. factor perfect square) } \\
\left(x+\frac{3}{2}\right)^{2} & =\frac{7}{4} & & \\
x+\frac{3}{2} & = \pm \sqrt{\frac{7}{4}} & & \text { (square root property) } \\
x & =-\frac{3}{2} \pm \frac{\sqrt{7}}{2} & &
\end{array}
$$

As you can see, you can now solve quadratic equations that would not be ones that you could factor using our previous factoring techniques. You can use the quadratic formula to solve quadratic equations that previously you solved by factoring.

EXAMPLE Solve $\frac{1}{15}+\frac{3}{y}=\frac{4}{y+1}$.

$$
\begin{array}{rlrl}
\frac{1}{15}+\frac{3}{y} & =\frac{4}{y+1} & & (\text { multiply by the LCD } 15 y(y+1)) \\
\frac{1}{15} \cdot 15 y(y+1)+\frac{3}{y} \cdot \mathbf{1 5 y}(\boldsymbol{y}+1) & =\frac{4}{y+1} \cdot \mathbf{1 5 y}(\boldsymbol{y}+1) & & \text { (cancel common factors) } \\
y(y+1)+45(y+1) & =60 y & & \text { (multiply out) } \\
y^{2}+y+45 y+45 & =60 y & & \text { (collect like terms) } \\
y^{2}-14 y+45 & =0 & & \text { (use quadratic formula) } \\
y & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} & & \\
y & =\frac{14 \pm \sqrt{(-14)^{2}-4(1)(45)}}{2(1)} & & \\
y & =\frac{14 \pm \sqrt{16}}{2} & & \\
& =\frac{14 \pm 4}{2} & & \\
& =7 \pm 2 \\
& =9 \text { or } y=5 &
\end{array}
$$

Since neither $y=9$ nor $y=5$ makes the LCD zero, these are both solutions. You could also substitute back into the original equation to determine that they are both solutions.

### 1.3 Pythagorean Theorem

The hypotenuse in a right triangle is the side opposite the 90 degree angle in the triangle.
Pythagorean theorem: $a^{2}=b^{2}+c^{2}$, where $a$ is the length of the hypotenuse in a right triangle and $b$ and $c$ are the lengths of the other two sides.


EXAMPLE A right triangle has one side of length 4 cm and hypotenuse of length 5 cm . What is the length of the other side?

$$
\begin{aligned}
a^{2} & =b^{2}+c^{2} \\
(5)^{2} & =(4)^{2}+c^{2} \\
9 & =c^{2} \quad \Rightarrow \quad c=3 \text { choose positive root }
\end{aligned}
$$

The other side has length 3 cm .

### 1.4 Transforming into a Quadratic

This is about using the mathematical concept of change of variables (sometimes called substitution) which is a powerful concept and will be useful in the future. All solutions should begin by clearly identifying the change of variables that converts the equation into a quadratic equation.
EXAMPLE Solve $\left(x^{2}+2 x\right)^{2}-\left(x^{2}+2 x\right)-12=0$.
Identify the substitution $\boldsymbol{u}=\boldsymbol{x}^{2}+\mathbf{x} \boldsymbol{x}$. The equation then becomes

$$
\begin{aligned}
\left(\boldsymbol{x}^{2}+\mathbf{x} \boldsymbol{x}\right)^{2}-\left(\boldsymbol{x}^{2}+\mathbf{2 x}\right)-12 & =0 & & \text { (substiutution) } \\
\boldsymbol{u}^{2}-\boldsymbol{u}-12 & =0 & & \\
(u+3)(u-4) & =0 & & \text { (factor) } \\
u+3 & =0 \text { or } u-4=0 & & \text { (zero factor property) } \\
u & =-3 \text { or } u=4 & &
\end{aligned}
$$

For each of these value of $u$ we get a quadratic in $x$ to solve:

$$
\begin{array}{rlr}
\boldsymbol{u}=\boldsymbol{x}^{2}+\mathbf{x} \boldsymbol{x} & =-3 \\
x^{2}+2 x+3 & =0 \\
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
x & =\frac{-2 \pm \sqrt{(2)^{2}-4(1)(3)}}{2(1)} \\
x & =\frac{-2 \pm \sqrt{-8}}{2} \\
x & =\frac{-2 \pm i \sqrt{8}}{2} \\
x & =\frac{-2 \pm i 2 \sqrt{2}}{2} \\
x & =-1 \pm i \sqrt{2} \\
\boldsymbol{u}=\boldsymbol{x}^{2}+\mathbf{2 x} & =4 \\
x^{2}+2 x-4 & =0 \\
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
x & =\frac{-2 \pm \sqrt{(2)^{2}-4(1)(-4)}}{2(1)} \\
x & =\frac{-2 \pm \sqrt{20}}{2} \\
x & =\frac{-2 \pm 2 \sqrt{5}}{2} \\
x & =-1 \pm \sqrt{5}
\end{array} \quad \text { (quadratic formula) } \quad \text { (quatic formula) }
$$

There are four solutions to the original equation. Two are complex valued ( $x=-1 \pm i \sqrt{2}$ ) , two are real valued $(x=-1 \pm \sqrt{5})$.

## 2 Sketching Quadratics

For a quadratic function $y=f(x)=a x^{2}+b x+c$, we can create a sketch by determining four things:

1. $x$-intercepts: determine these by using the quadratic formula $x_{\text {intercept }}=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$.

NOTE: If $b^{2}-4 a c<0$ then there are no $x$-intercepts (they are not real numbers).
2. Vertex: $\left(x_{\text {Vertex }}, y_{\text {Vertex }}\right)=\left(-\frac{b}{2 a}, f\left(-\frac{b}{2 a}\right)\right)$.

$$
\begin{aligned}
& x_{\text {Vertex }}=-\frac{b}{2 a} \\
& y_{\text {Vertex }}=f\left(-\frac{b}{2 a}\right)=-\frac{b^{2}-4 a c}{4 a}
\end{aligned}
$$

How to remember this: The $x$-coordinate of the vertex will be right in the middle of the two $x$-intercepts, even if the $x$-intercepts are not real numbers!

$$
\begin{aligned}
x_{\text {Vertx }} & =\frac{1}{2}\left(\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}+\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}\right) \\
& =\frac{1}{2}\left(\frac{-b+\sqrt{b^{2}-4 a c}-b-\sqrt{b^{2}-4 a c}}{2 a}\right) \\
& =\frac{1}{2}\left(\frac{-2 b}{2 a}\right) \\
& =-\frac{b}{2 a}
\end{aligned}
$$

3. $y$-intercept: evaluate at $x=0$, so figure out what $f(0)$ is.
4. The function opens up if $a>0$, and opens down if $a<0$.

All this boils down nicely if you look at the completing the square version of the quadratic:

$$
\begin{aligned}
f(x) & =a x^{2}+b x+c & & \text { (standard form) } \\
& =a\left[x+\frac{b}{2 a}\right]^{2}-\frac{b^{2}-4 a c}{4 a} & & \text { (from completing the square) } \\
& =a[x-\boldsymbol{h}]^{2}+\boldsymbol{k} & & \text { (vertex form) }
\end{aligned}
$$

The final form is called the vertex form since the vertex of the quadratic is $(h, k)$. Let's verify that. The smallest or largest value of the quadratic will be where $x_{\text {Vertex }}=-b /(2 a)=h$, since

$$
f(\boldsymbol{h})=a[\boldsymbol{b}-\hbar]^{2^{2}}+k=k=-\frac{b^{2}-4 a c}{4 a}=y_{\mathrm{Vertex}}
$$

## Sketching Quadratic in standard form $f(x)=a x^{2}+b x+c$

1. Determine any $x$-intercepts (if they exist) by solving $f(x)=0$,
2. Determine the vertex using $x_{\text {Vertex }}=-b /(2 a)$ and $y_{\text {Vertex }}=f\left(x_{\text {Vertex }}\right)$.
3. Determine the $y$-intercept, $f(0)$.
4. Determine if it opens up $(a>0)$ or down $(a<0)$,

## Sketching Quadratic in vertex form form $f(x)=a(x-h)^{2}+k$

1. Determine any $x$-intercepts (if they exist) by solving $f(x)=0$,
2. Determine the vertex using $x_{\text {Vertex }}=h$ and $y_{\text {Vertex }}=k$.
3. Determine the $y$-intercept, $f(0)$.
4. Determine if it opens up $(a>0)$ or down $(a<0)$,

EXAMPLE Sketch the parabola $y=5 x^{2}+4 x-12$. Label the vertex, $y$-intercept, and any $x$-intercepts on your sketch.

To get the $x$-intercepts, use the quadratic formula.
In this case, $a=5, b=4$, and $c=-12$.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-4 \pm \sqrt{4^{2}-4(5)(-12)}}{2(5)} \\
& =\frac{-4 \pm \sqrt{256}}{10} \\
& =\frac{-4 \pm 16}{10} \\
& =\frac{-4+16}{10} \text { or } \frac{-4-16}{10} \\
& =\frac{12}{10} \text { or } \frac{-20}{10} \\
& =\frac{6}{5} \text { or }-2
\end{aligned}
$$

The vertex is located at:

$$
\begin{aligned}
x & =\frac{-b}{2 a}=\frac{-4}{2(5)}=-\frac{2}{5} \\
y & =f\left(-\frac{2}{5}\right) \\
& =5\left(-\frac{2}{5}\right)^{2}+4\left(-\frac{2}{5}\right)-12 \\
& =-\frac{64}{5}
\end{aligned}
$$

The vertex is at $\left(-\frac{2}{5},-\frac{64}{5}\right)$.

To get the $y$-intercept, evaluate $f(0)$ :

$$
y=f(0)=5(0)^{2}+4(0)-12=-12
$$

A quadratic opens up if $a>0$, and opens down if $a<0$. Since $a=5$ in this case, this quadratic opens up.

You can now put this all together to get the sketch:


EXAMPLE What is the domain of the function $f(x)=\sqrt{x^{2}-3 x+1}$ ? Sketch your answer on a number line.
Domain: what real numbers can I put into this expression and get a real number out?

$$
x^{2}-3 x+1 \geq 0
$$

We only know how to algebraically solve linear inequalities, but we can sketch $y=x^{2}-3 x+1$ and get the answer from our sketch by figuring out where $x^{2}-3 x+1$ is positive.
Since $a=1>0$, this quadratic opens up.
To get the $x$-intercepts, use the quadratic formula, $a=1, b=-3$, and $c=1$ :

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{3 \pm \sqrt{(-3)^{2}-4(1)(1)}}{2(1)} \\
& =\frac{3 \pm \sqrt{5}}{2} \\
& =\frac{3+\sqrt{5}}{2} \text { or } \frac{3-\sqrt{5}}{2}
\end{aligned}
$$

Then the vertex is located at:

$$
\begin{aligned}
& x=\frac{-b}{2 a}=\frac{-(-3)}{2(1)}=\frac{3}{2} \\
& y=\left(\frac{3}{2}\right)^{2}-3\left(\frac{3}{2}\right)+1=\frac{9}{4}-\frac{9}{2}+1=\frac{9}{4}-\frac{18}{4}+\frac{4}{4}=\frac{-5}{4}
\end{aligned}
$$

The vertex is at $\left(\frac{3}{2},-\frac{5}{4}\right)$.
We don't need the $y$-intercept to answer the question. You can now put this all together to get the sketch:


