End Behaviour

"End behaviour" is an attempt to determine what a function $f(x)$ is doing when $|x|$ is "large". Large is somewhat vague, but means out past any zeros, any vertical asymptotes, or any holes.

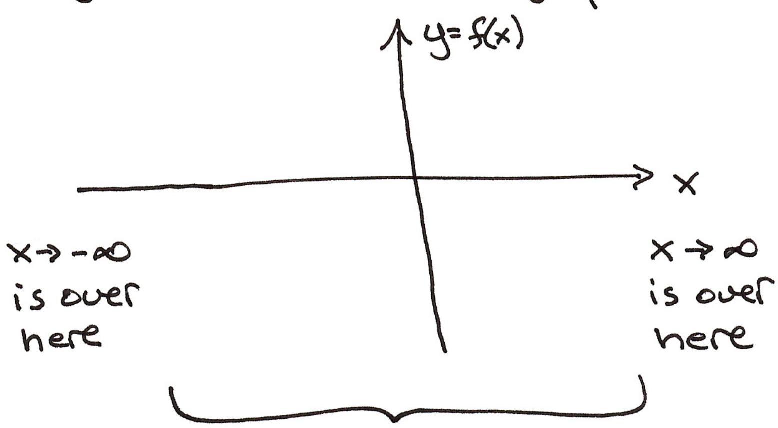
We can express this mathematically (and hence more precisely) with the notation

$\lim_{x \to \infty} [f(x)]$ which we read as "the limit as x approaches infinity of $f(x)$".

and

$\lim_{x \to -\infty} [f(x)]$ which we read as "the limit as x approaches minus infinity of $f(x)$".

On a graph, this means we are interested in the far left $(x \to -\infty)$ and far right $(x \to \infty)$ of the graph.



End Behaviour analysis <u>does not</u> tell you anything about what $f(x)$ is doing in the middle.

End Behaviour of Polynomials

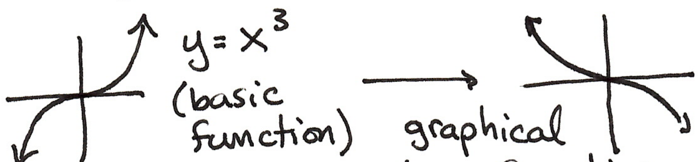
Polynomials will approach either $+\infty$ or $-\infty$ as $|x| \to \infty$. Since polynomials are dominated by their leading term, we can use the leading term to determine the end behaviour.

Ex] $f(x) = \underbrace{-7x^3}_{\text{leading term.}} + 3X - 2.$

$$\lim_{x\to\infty}\left[f(x)\right] \sim \lim_{x\to\infty}\left[-7x^3\right] = -\infty$$

$$\lim_{x\to-\infty}\left[f(x)\right] \sim \lim_{x\to-\infty}\left[-7x^3\right] = \infty$$

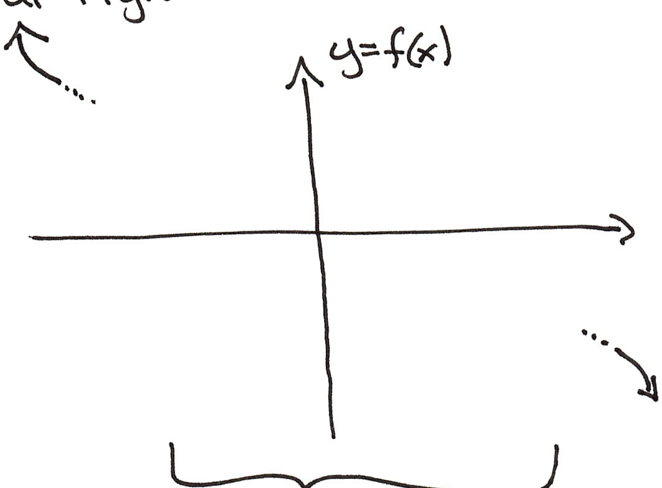
To figure out the end behaviour, we can draw a quick sketch of $y=-7x^3$:



graphical transformation
reflect over x-axis
(and vertical stretch by factor of 7, but that isn't important for end behaviour).

from this sketch we see

This tells us what f(x) looks like on the far left and far right:



To figure out what is happening in middle, we need to look for zeros.

End behaviour of Rational Functions

Rational functions have more possibilities than polynomials.

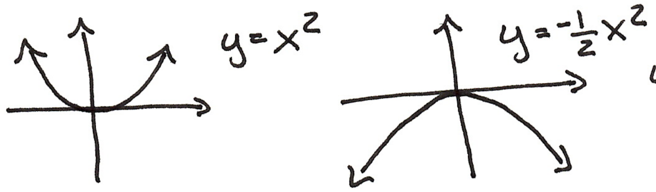
① If degree of numerator is larger than degree of denominator (by 2 or more!), the behaviour is like polynomials.

Ex] $f(x) = \dfrac{3x^4 + 7x - 1}{-6x^2 + 2x - 1}$ Each polynomial is dominated by its leading term.

$$\lim_{x \to \infty} [f(x)] \sim \lim_{x \to \infty} \left[\frac{3x^4}{-6x^2}\right] = \lim_{x \to \infty} \left[-\frac{1}{2}x^2\right] = -\infty$$

$$\lim_{x \to -\infty} [f(x)] \sim \lim_{x \to \infty} \left[-\frac{1}{2}x^2\right] = -\infty$$

sketch



used to get

② If degree of numerator is one more than degree of denominator, we get a slant (or oblique) asymptote:

Ex] $f(x) = \dfrac{4x^3 + 2x^2 + x}{2x^2 + 2x - 1}$

linear → slant asymptote.

$$\lim_{x \to \infty} [f(x)] \sim \lim_{x \to \infty} \left[\frac{4x^3}{2x^2}\right] = \lim_{x \to \infty} [2x] = \infty$$

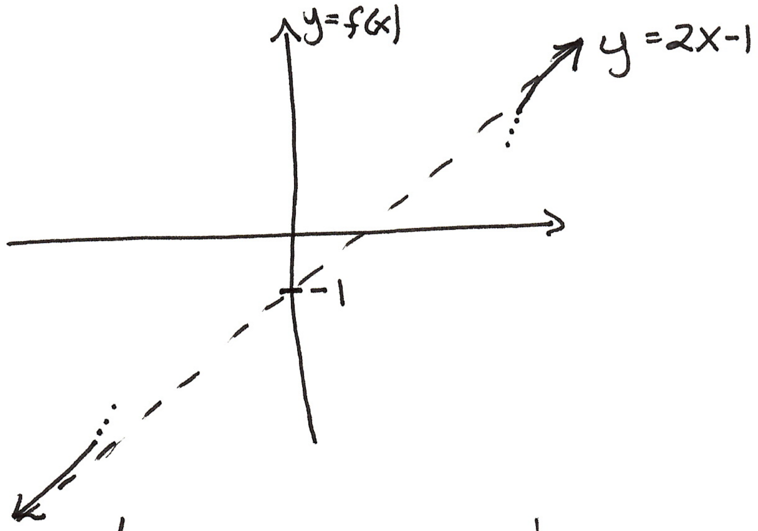
$$\lim_{x \to -\infty} [f(x)] \sim \lim_{x \to -\infty} [2x] = -\infty$$

To get equation of slant asymptote, divide ~~numerator~~ denominator into ~~denominato~~ numerator (whoops!)

$$\begin{array}{r|l} & 2x - 1 \\ 2x^2 + 2x - 1 & 4x^3 + 2x^2 + x + 0 \\ & \underline{4x^3 + 4x^2 - 2x} \\ & -2x^2 + 3x + 0 \\ & \underline{-2x^2 - 2x + 1} \\ & 5x - 1 \text{ (remainder)} \end{array}$$

$$\rightarrow f(x) = \underbrace{2x - 1}_{} + \frac{5x - 1}{2x^2 + 2x - 1}$$

↳ Equation of slant asymptote is $y = 2x - 1$.



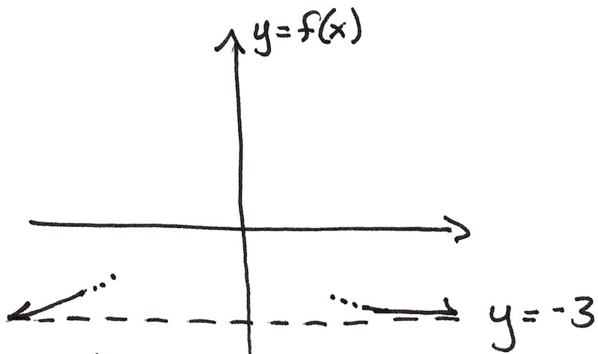
To figure out what is happening in middle, we need to look at zeros, vertical asymptotes, and holes.

③ If degree of numerator is equal to degree of denominator, we get a horizontal asymptote.

Ex) $f(x) = \dfrac{12x^3 - 7x + 2}{-4x^3 + 17}$

$$\lim_{x\to\infty}[f(x)] \sim \lim_{x\to\infty}\left[\frac{12x^3}{-4x^3}\right] = \lim_{x\to\infty}[-3] = -3$$

$$\lim_{x\to-\infty}[f(x)] \sim \lim_{x\to-\infty}[-3] = -3$$



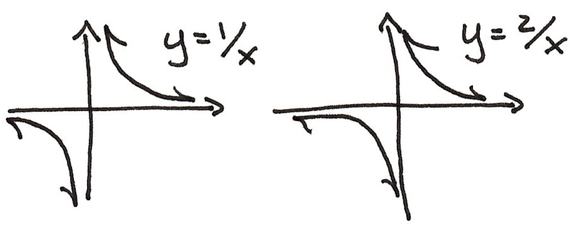
To figure out what is happening in middle, we need to look at zeros, vertical asymptotes, and holes.

④ If degree of numerator is less than degree of denominator, we have a horizontal asymptote of $y=0$.

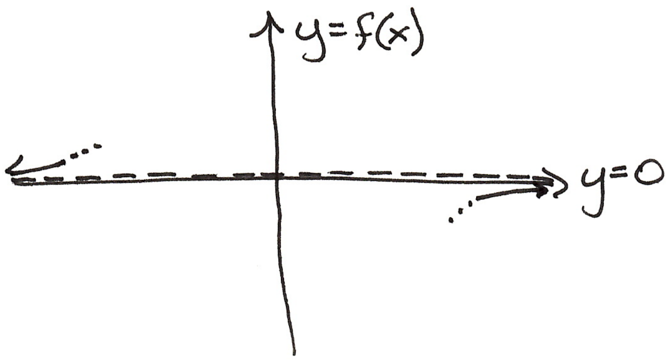
Ex $f(x) = \dfrac{14x^2+7}{7x^3+2x-3}$

$$\lim_{x\to\infty}\left[f(x)\right] \sim \lim_{x\to\infty}\left[\frac{14x^2}{7x^3}\right] = \lim_{x\to\infty}\left[\frac{2}{x}\right] = 0$$

$$\lim_{x\to-\infty}\left[f(x)\right] \sim \lim_{x\to-\infty}\left[\frac{2}{x}\right] = 0$$



used to get



An Alternative to Considering Dominance of Leading Term

For rational functions, we can divide everything by highest power of x in denominator. This is a more mathematically precise way to ~~do it~~ evaluate limits, and what you will do in calculus. You use the fact that

$$\lim_{x\to\pm\infty}\left[\frac{1}{x^n}\right] = 0 \quad \text{for } n>0$$

to simplify the limits.

Let's redo all the limits for rational functions in the examples using these ideas.

Ex $f(x) = \dfrac{3x^4+7x-1}{-6x^2+2x-1} = \dfrac{3x^2+\frac{7}{x}-\frac{1}{x^2}}{-6+\frac{2}{x}-\frac{1}{x^2}}$

$$\lim_{x\to\infty}\left[f(x)\right] = \lim_{x\to\infty}\left[\frac{3x^2+\frac{7}{x}-\frac{1}{x^2}}{-6+\frac{2}{x}-\frac{1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{3x^2+0-0}{-6+0-0}\right]$$

$$= \lim_{x\to\infty}\left[-\frac{1}{2}x^2\right]$$

$$= -\infty$$

Note there is no approximation in this computation!

$$\lim_{x\to-\infty}\left[f(x)\right] = \lim_{x\to-\infty}\left[-\frac{1}{2}x^2\right] = -\infty.$$

Ex $f(x) = \dfrac{4x^3+2x^2+x}{2x^2+2x-1} = \dfrac{4x+2+\frac{1}{x}}{2+\frac{2}{x}-\frac{1}{x^2}}$

$$\lim_{x\to\infty}\left[f(x)\right] = \lim_{x\to\infty}\left[\frac{4x+2+\frac{1}{x}}{2+\frac{2}{x}-\frac{1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{4x+2+0}{2+0-0}\right]$$

$$= \lim_{x\to\infty}\left[2x+1\right]$$

$$= \infty$$

$$\lim_{x\to-\infty}\left[f(x)\right] = \lim_{x\to-\infty}\left[2x+1\right] = -\infty.$$

Ex $f(x) = \dfrac{12x^3-7x+2}{-4x^3+17} = \dfrac{12-\frac{7}{x^2}+\frac{2}{x^3}}{-4+\frac{17}{x^3}}$

$$\lim_{x\to\infty}\left[f(x)\right] = \lim_{x\to\infty}\left[f(x)\right] = \lim_{x\to\infty}\left[\frac{12-\frac{7}{x^2}+\frac{2}{x^3}}{-4+\frac{17}{x^3}}\right] = \lim_{x\to\infty}\left[\frac{12}{-4}\right] = -3$$

$$\lim_{x\to-\infty}\left[f(x)\right] = \lim_{x\to-\infty}\left[\frac{12}{-4}\right] = -3$$

Ex $f(x) = \dfrac{14x^2+7}{7x^3+2x-3} = \dfrac{\frac{14}{x} + \frac{7}{x^3}}{7 + \frac{2}{x^2} - \frac{3}{x^3}}$

$$\lim_{x\to\infty}\left[f(x)\right] = \lim_{x\to\infty}\left[\frac{\frac{14}{x} + \frac{7}{x^3}}{7 + \frac{2}{x^2} - \frac{3}{x^3}}\right]$$

$$= \frac{0+0}{7+0-0} = \frac{0}{7} = 0.$$

$$\lim_{x\to-\infty}\left[f(x)\right] = \frac{0}{7} = 0.$$