Questions

1. Find all the real and imaginary roots of the equation, including multiplicity

$$-3 + 7x - 3x^2 - 3x^3 + 2x^4 = 0$$

2. Find all the real and imaginary roots of the equation, including multiplicity

$$4x^4 - 12x^3 + 13x^2 - 6x + 1 = 0$$

3. Find a polynomial with real valued coefficients that has the following roots: $\frac{5}{4}$ with multiplicity 2, and $2 - 3i$.

4. Use Descartes's Rule of Signs to discuss the possibilities for the roots of the equation

$$t^4 - 3t^3 + 2t^2 - 5t + 7 = 0.$$

5. Use Descartes's Rule of Signs to discuss the possibilities for the roots of the equation

$$x^5 + x^3 + 5x = 0.$$

6. Show that $f(x) = x^3 - 64$ has one real zero of multiplicity 1.

① $f(x) = -3 + 7x - 3x^2 - 3x^3 + 2x^4 = 0$

$\Rightarrow 2x^4 - 3x^3 - 3x^2 + 7x - 3 = 0$

factors of -3: $\pm(1,3)$

factors of 2: $\pm(1,2)$.

Possible Rational zeros: $\pm\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)$.

$f(1) = 0 \rightarrow x-1$ is a factor.

$$
\begin{array}{rl}
 & 2x^3 - x^2 - 4x + 3 \\
x-1 \,\big) & 2x^4 - 3x^3 - 3x^2 + 7x - 3 \\
 & \underline{2x^4 - 2x^3} \qquad \text{subtract} \\
 & -x^3 - 3x^2 \\
 & \underline{-x^3 + x^2} \qquad \text{subtract} \\
 & -4x^2 + 7x \\
 & \underline{-4x^2 + 4x} \qquad \text{subtract} \\
 & 3x - 3 \\
 & \underline{3x - 3} \qquad \text{subtract} \\
 & 0 \qquad \text{(remainder is zero)}
\end{array}
$$

So $f(x) = (x-1)(2x^3 - x^2 - 4x + 3)$

Now work on $g(x) = 2x^3 - x^2 - 4x + 3$

factors of 3: $\pm(1,3)$

factors of 2: $\pm(1,2)$

Possible Rational Zeros: $\pm\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)$.

$g(1) = 0$ so $x-1$ is a factor.

$$
\begin{array}{r|l}
 & \;\;2x^2 + x - 3 \\
\hline
x-1 & 2x^3 - x^2 - 4x + 3 \\
 & \underline{2x^3 - 2x^2} \qquad\qquad \text{subtract} \\
 & \qquad x^2 - 4x \\
 & \qquad \underline{x^2 - x} \qquad\qquad \text{subtract} \\
 & \qquad\quad -3x + 3 \\
 & \qquad\quad \underline{-3x + 3} \qquad \text{subtract} \\
 & \qquad\qquad\quad 0 \qquad \text{(remainder is zero)}
\end{array}
$$

So $g(x) = (x-1)(2x^2+x-3)$

Factor $2x^2+x-3 = (x-1)(2x+3)$

So $f(x) = (x-1)\,g(x)$

$$= (x-1)(x-1)(x-1)(2x+3)$$

$$= (x-1)^3(2x+3)^1$$

zeros $x = 1$ multiplicity 3

$x = -3/2$ multiplicity 1.

(2) $f(x) = 4x^4 - 12x^3 + 13x^2 - 6x + 1 = 0$

factors of 1: ± 1

factors of 4: $\pm(1, 2, 4)$

Possible Rational zeros: $\pm\left(1, \frac{1}{2}, \frac{1}{4}\right)$.

$f(1) = 0$ so $x-1$ is a factor.

$$
\begin{array}{r l}
 & 4x^3 - 8x^2 + 5x - 1 \\
x-1 \,\big) & 4x^4 - 12x^3 + 13x^2 - 6x + 1 \\
 & \underline{4x^4 - 4x^3} \qquad \text{subtract} \\
 & -8x^3 + 13x^2 \\
 & \underline{-8x^3 + 8x^2} \qquad \text{subtract} \\
 & 5x^2 - 6x \\
 & \underline{5x^2 - 5x} \qquad \text{subtract} \\
 & -x + 1 \\
 & \underline{-x + 1} \qquad \text{subtract} \\
 & 0 \qquad \text{(remainder is zero).}
\end{array}
$$

so $f(x) = (x-1)(4x^3 - 8x^2 + 5x - 1)$

Now work on $g(x) = 4x^3 - 8x^2 + 5x - 1$

factors of -1: ± 1

factors of 4: $\pm(1, 2, 4)$

Possible Rational zeros: $\pm\left(1, \frac{1}{2}, \frac{1}{4}\right)$

$g(1) = 0$ so $x-1$ is a factor.

$$\begin{array}{r}
4x^2 - 4x + 1 \\
x-1 \overline{\smash{)}\, 4x^3 - 8x^2 + 5x - 1} \\
\underline{4x^3 - 4x^2} \quad \text{subtract} \\
-4x^2 + 5x \\
\underline{-4x^2 + 4x} \quad \text{subtract} \\
x - 1 \\
\underline{x - 1} \quad \text{subtract} \\
0 \quad \text{(remainder is zero)}
\end{array}$$

so $g(x) = (x-1)(4x^2-4x+1)$
$= (x-1)(2x-1)^2$

so $f(x) = (x-1)\,g(x)$
$= (x-1)^2(2x-1)^2$

<u>Zeros</u>: $x=1$ multiplicity 2
$x=\frac{1}{2}$ multiplicity 2.

③ Polynomial with real valued coefficients that has a complex root $2-3i$ will also have root $2+3i$ since complex roots will appear in complex conjugate pairs.

$$P(x) = (x - 5/4)^2 (x - (2-3i))(x - (2+3i))$$

Note $(x - 5/4)^2 = \left(\frac{1}{4}(4x-5)\right)^2 = \frac{1}{16}(4x-5)^2$.

We could also use just $(4x-5)^2$, since that would have a zero of 5/4 of multipicity 2.

new P(x), not equal to one above. Let's call it Q(x) to avoid confusion.

$$\cancel{P(x)}\ Q(x) = (4x-5)^2 (x - (2-3i))(x - (2+3i))$$

Multiply it out to get polynomial.

$$(4x-5)^2 = 16x^2 - 40x + 25$$

$$(x-2+3i)(x-2-3i) = x^2 - 2x - \cancel{3xi} - 2x + 4 + 6i + \cancel{3xi} - 6i - 9i^2$$

$$= x^2 - 4x + 4 - 9(-1) \quad \text{use } i^2 = -1$$

$$= x^2 - 4x + 13$$

$$Q(x) = (16x^2 - 40x + 25)(x^2 - 4x + 13)$$

$$= 16x^4 - 64x^3 + \overset{208}{\cancel{64}}x^2 - 40x^3 + 160x^2 - 520x + 25x^2 - 100x + 325$$

$$= 16x^4 - 104x^3 + \underset{393}{\cancel{154}}x^2 - 620x + 325.$$

④ $P(t) = t^4 - 3t^3 + 2t^2 - 5t + 7 \cancel{=0}$

4 variations of sign.

$P(-t) = (-t)^4 - 3(-t)^3 + 2(-t)^2 - 5(-t) + 7 \cancel{=0}$

$= t^4 + 3t^3 + 2t^2 + 5t + 7$

no variations of sign.

The number of positive real roots is 4, 2, or 0.
The number of negative real roots is 0.

Possibilities:

- 4 positive real roots
- 2 positive real roots, 2 complex roots (complex conjugate pairs)
- 4 complex roots, appearing in 2 sets of complex conjugate pairs.

⑤ $P(x) = x^5 + x^3 + 5x = 0$

Note $x=0$ is ~~always~~ a root.

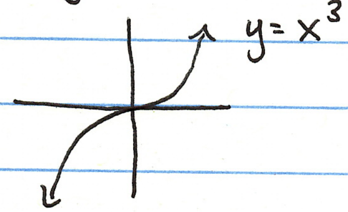
$P(x) = x^5 + x^3 + 5x$ No variations of sign
$P(-x) = -x^5 - x^3 - 5x$ No variations of sign.

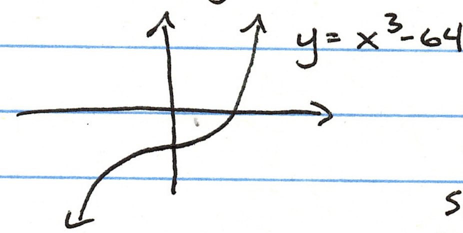
The number of positive real roots is zero.
The number of negative real roots is zero.

Since $x=0$ is ~~always~~ a root, the remaining 4 roots must be complex, appearing in 2 sets of complex conjugate pairs.

⑥ Here are three different ways to show this.

ⓐ Notice $y=x^3-64$ is sketch of $y=x^3$ moved down 64 units.





ONE real root. Multiplicity is 1 since graph is not horizontal at root.

$$x^3-64=0$$
$$x=64^{1/3}=4.$$

ⓑ Notice x^3-64 evaluated at $x=4$ is zero, so $x-4$ is a factor.

$$\begin{array}{r l}
 & x^2+4x+16 \\
x-4 \,\big) & x^3+0x^2+0x-64 \\
 & \underline{x^3-4x^2} \qquad \text{subtract} \\
 & \quad 4x^2+0x \\
 & \quad \underline{4x^2-16x} \qquad \text{subtract} \\
 & \qquad 16x-64 \\
 & \qquad \underline{16x-64} \qquad \text{subtract} \\
 & \qquad\qquad 0
\end{array}$$

$$\rightarrow x^3-64=(x-4)\underbrace{(x^2+4x+16)}_{\text{irreducible.}}$$

ⓒ Use memorized formula for difference of cubes

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$x^3-64=x^3-4^3 \qquad a=x, \quad b=4$$

$$x^3-4^3=(x-4)\underbrace{(x^2+4x+16)}_{\text{irreducible.}}$$

Note: $x^2+4x+16$ has no real-valued roots (irreducible on reals)

$$x=\frac{-4\pm\sqrt{16-4(1)(16)}}{2(1)}$$