Questions

Sketch the following polynomials, by determining:

- x-intercepts with multiplicity
- end behaviour (show the end behaviour by determining $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

1. $f(x) = (x-1)^3(3x+2)^3x$.

2. $f(x) = (x-2)^2(1-x)^2(7x-9)$.

3. $f(x) = 4x^3 - 3x + 1$.

4. $f(x) = x(x+6)(x^2 - x + 12)$.

Solve the following polynomial inequalities using a sign chart.

5. $x^2 - x - 12 < 0$.

6. $x^3 - 3x > 0$.

7. $-x^3 + 3x + 2 < 0$.

8. $x^3 + 2x^2 - 2x - 4 < 0$.

9. $x^4 - 5x^3 + 3x^2 + 15x - 18 \geq 0$.

① $f(x) = (x-1)^3(3x+2)^3x$

zeros

$x = 1$, multiplicity 3 (odd) f changes sign.

$x = -2/3$, multiplicity 3 (odd) f changes sign.

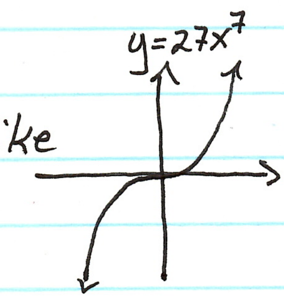
mult > 1, so f will be horizontal at these zeros.

$x = 0$, multiplicity 1 (odd) f changes sign.

End behaviour

If $|x|$ is large, $f(x) \sim (x)^3(3x)^3x$

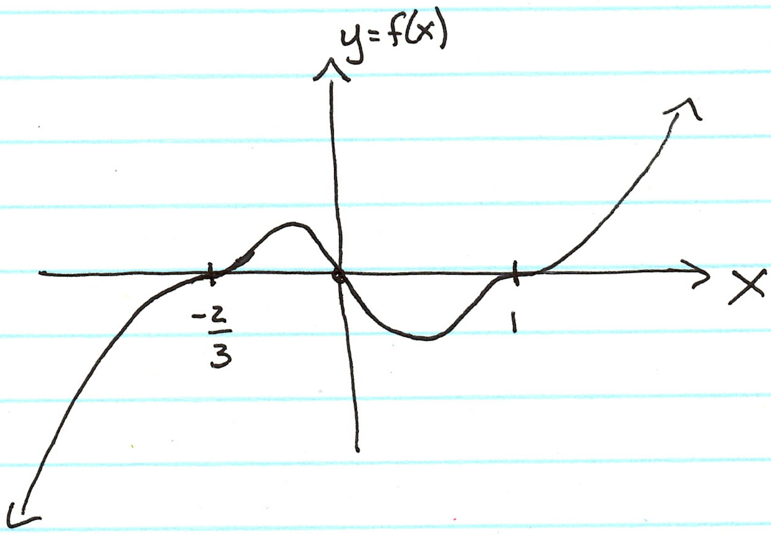
$\sim 27x^7$ which looks like



∴ Therefore,

$$\lim_{x\to\infty} f(x) = \infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$



(2) $f(x) = (x-2)^2(1-x)^2(7x-9)$

zeros $x=2$ multiplicity 2 (even) f does not change sign

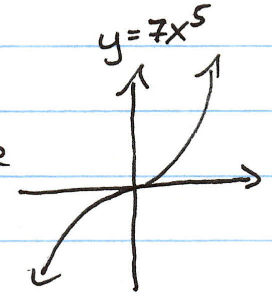
$x=1$ multiplicity 2 (even) f does not change sign.

$x=9/7$ multiplicity 1 (odd) f changes sign.

End behaviour

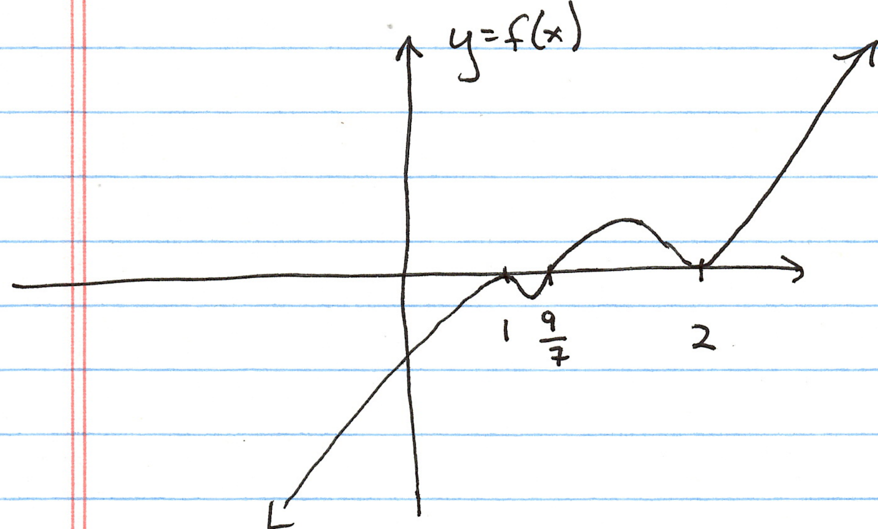
If $|x|$ is large, $f(x) \sim (x)^2(-x)^2(7x)$

$\sim 7x^5$ which looks like



Therefore $\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to -\infty} f(x) = -\infty.$



③ $f(x) = 4x^3 - 3x + 1$ factor

factors of 1: ± 1
factors of 4: $\pm(1, 2, 4)$
Potential Rational zeros: $\pm\left(1, \frac{1}{2}, \frac{1}{4}\right)$.

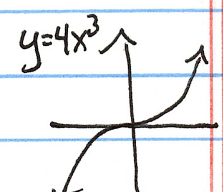
$f(-1) = 4(-1)^3 - 3(-1) + 1 = 0$ $x+1$ is a factor.

$$\begin{array}{r l}
 & 4x^2 - 4x + 1 \\
x+1 \,\big)\!\! & \overline{4x^3 + 0x^2 - 3x + 1} \\
 & \underline{4x^3 + 4x^2} \qquad \text{subtract} \\
 & \quad -4x^2 - 3x \\
 & \quad \underline{-4x^2 - 4x} \qquad \text{subtract} \\
 & \qquad\quad x + 1 \\
 & \qquad\quad \underline{x + 1} \qquad \text{subtract} \\
 & \qquad\qquad 0 \qquad \text{(remainder is zero)}
\end{array}$$

$f(x) = (x+1)(4x^2 - 4x + 1)$
$\quad\; = (x+1)(2x-1)^2$

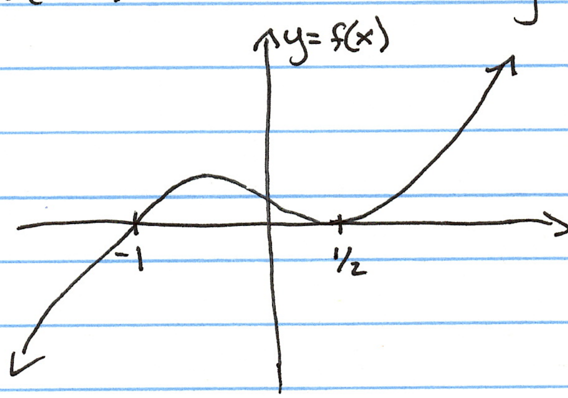
Zeros $x = -1$ multiplicity 1 (odd) so f changes sign.
$x = \frac{1}{2}$ multiplicity 2 (even) so f does not change sign.

End behaviour:
$f(x) \sim 4x^3$ if $|x|$ large



$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$



(4) $f(x) = x(x+6)(x^2-x+12)$

~~$= x(x+6)(x$~~

x^2-x+12 might be irreducible (no real roots)

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

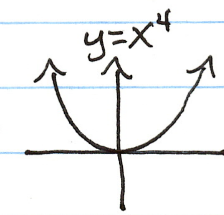
$$= \frac{1 \pm \sqrt{1-4(1)(12)}}{2}$$ no Real roots.

<u>zeros</u> $\left.\begin{array}{l} x=0 \\ x=-6 \end{array}\right\}$ multiplicity 1 (odd) so f will change sign.

<u>End behaviour</u>

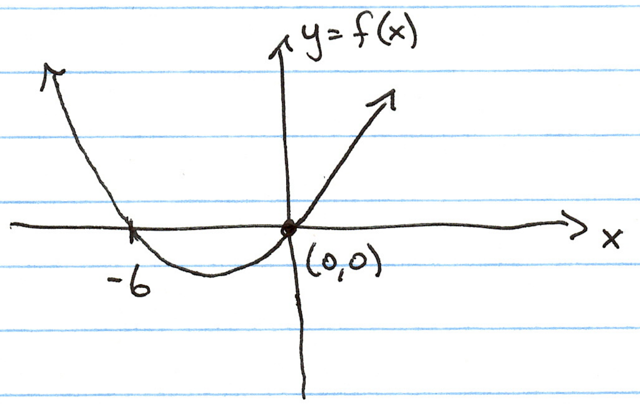
If $|x|$ large, $f(x) \sim x(x)(x^2)$

$= x^4$ which looks like



Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$

$\lim\limits_{x\to-\infty} f(x) = \infty$



(5) $x^2 - x - 12 < 0$

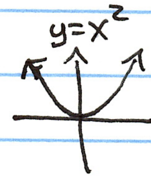
$f(x) = x^2 - x - 12$

$\quad = (x+3)(x-4)$

<u>zeros</u> $\left.\begin{array}{l} x=3 \\ x=-4 \end{array}\right\}$ multiplicity 1, so f changes sign.

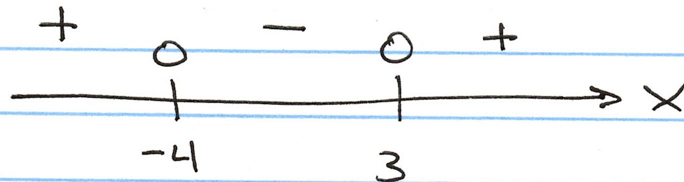
<u>End behaviour</u>

If $|x|$ large, $f(x) \sim x^2$



So $\lim_{x\to\infty} f(x) = \infty$ (positive)

$\lim_{x\to-\infty} f(x) = \infty$ (positive)



So $f(x) < 0$ for $x \in (-4, 3)$.

(6) $x^3 - 3x > 0$

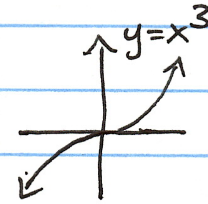
$x(x^2 - 3) > 0$

$f(x) = x(x - \sqrt{3})(x + \sqrt{3}) > 0$

<u>zeros</u> $\left.\begin{array}{l} x=0 \\ x=\sqrt{3} \\ x=-\sqrt{3} \end{array}\right\}$ multiplicity 1, so f will change sign

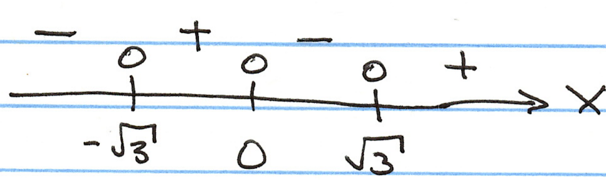
<u>End behaviour</u>

If $|x|$ is large, $f(x) \sim x^3$



so $\lim_{x\to\infty} f(x) = \infty$ (positive)

$\lim_{x\to-\infty} f(x) = -\infty$ (neg).



so $f(x) > 0$ for $x \in (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.

⑦ $f(x) = -x^3+3x+2 < 0$

factors of 2: $\pm(1,2)$

factors of -1: ± 1

Potential Rational Factors $\pm(1,2)$

$f(-1) = -(-1)^3 + 3(-1)+2 = 1-3+2=0 \rightarrow x+1$ is a factor.

$$\begin{array}{r l}
 & -x^2+x+2 \\
x+1 \overline{)} & -x^3+0x^2+3x+2 \\
 & \underline{-x^3-x^2} \qquad \text{subtract} \\
 & \quad x^2+3x \\
 & \quad \underline{x^2+x} \qquad \text{subtract} \\
 & \qquad 2x+2 \\
 & \qquad \underline{2x+2} \qquad \text{subtract} \\
 & \qquad\quad 0 \qquad \text{(remainder is zero)}
\end{array}$$

$f(x) = (x+1)(-x^2+x+2)$

$= -(x+1)(x^2-x-2)$

$= -(x+1)(x-2)(x+1)$

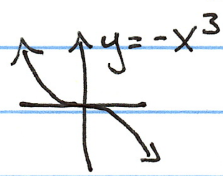
$= -(x+1)^2(x-2)$

zeros $x=-1$ multiplicity 2, so f does not change sign.

$x=2$ multiplicity 1, so f changes sign.

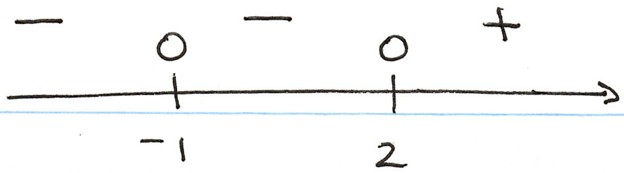
End behaviour

If $|x|$ large, $f(x) \sim -x^3$



$\lim_{x\to\infty} f(x) = -\infty$

$\lim_{x\to-\infty} f(x) = \infty$.



So $f(x)<0$ when $x \in (-\infty,-1) \cup (-1,2)$.

(8) $f(x) = x^3+2x^2-2x-4<0$

factors of -4: $\pm(1,2,4)$

factors of 1: $\pm(1)$

Possible Rational Factors: $\pm(1,2,4)$

$f(-1) = (-1)^3+2(-1)^2-2(-1)-4 = -1+2+2-4 \neq 0$.

$f(2) = (2)^3+2(2)^2-2(2)-4 = 8+8-4-4 \neq 0$

$f(-2) = (-2)^3+2(-2)^2-2(-2)-4 = -8+8+4-4=0 \to x+2$ will factor.

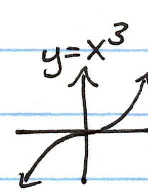
$$
\begin{array}{r l}
 & x^2-2 \\
x+2 \,\big) & x^3+2x^2-2x-4 \\
 & \underline{x^3+2x^2} \quad \text{subtract} \\
 & \quad -2x-4 \\
 & \quad \underline{-2x-4} \quad \text{subtract} \\
 & \qquad 0 \quad \text{(remainder is zero).}
\end{array}
$$

$f(x) = (x+2)(x^2-2) = (x+2)(x-\sqrt{2})(x+\sqrt{2})$

Zeros
$$\left.\begin{array}{l} x=-2 \\ x=\sqrt{2} \\ x=-\sqrt{2} \end{array}\right\}$$ multiplicity 1, so f will change sign.

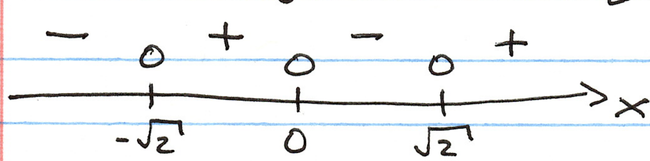
End behaviour

If $|x|$ is large, $f(x) \sim x^3$



so $\lim_{x\to\infty} f(x)=\infty$

$\lim_{x\to-\infty} f(x)=-\infty$



So $f(x)<0$ when $x \in (-\infty,-\sqrt{2}) \cup (0,\sqrt{2})$

⑨ $f(x) = x^4 - 5x^3 + 3x^2 + 15x - 18 \geq 0$

factors of -18: $\pm(1, 2, 3, 6, 9, 18)$

factors of 1: ± 1

Possible Rational Factors: $\pm(1, 2, 3, 6, 9, 18)$

$f(2) = (2)^4 - 5(2)^3 + 3(2)^2 + 15(2) - 18 = 16 - 40 + 12 + 30 - 18 = 0$

so $x - 2$ is a factor.

$$
\begin{array}{rl}
 & x^3 - 3x^2 - 3x + 9 \\
x - 2 \,\big)\!\! & x^4 - 5x^3 + 3x^2 + 15x - 18 \\
 & \underline{x^4 - 2x^3} \qquad \text{subtract} \\
 & -3x^3 + 3x^2 \\
 & \underline{-3x^2 + 6x^2} \qquad \text{subtract} \\
 & -3x^2 + 15x \\
 & \underline{-3x^2 + 6x} \qquad \text{subtract} \\
 & 9x - 18 \\
 & \underline{9x - 18} \qquad \text{subtract} \\
 & 0 \qquad \text{(remainder is zero)}
\end{array}
$$

$f(x) = (x-2)(x^3 - 3x^2 - 3x + 9)$

Now factor $x^3 - 3x^2 - 3x + 9 = g(x)$

factors of 9: $\pm(1, 3, 9)$

factors of 1: ± 1

Possible Rational factors $\pm(1, 3, 9)$

$g(3) = (3)^3 - 3(3)^2 - 3(3) + 9 = 27 - 27 - 9 + 9 = 0.$

so $x - 3$ is a factor.

$$
\begin{array}{r l}
 & x^2 \qquad -3 \\
x-3 \,\big) & x^3-3x^2-3x+9 \\
 & \underline{x^3-3x^2} \qquad \text{subtract} \\
 & \qquad -3x+9 \\
 & \qquad \underline{-3x+9} \quad \text{subtract.} \\
 & \qquad\quad 0
\end{array}
$$

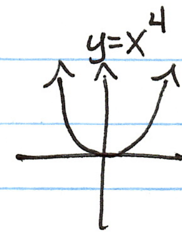
so $f(x) = (x-2)(x-3)(x^2-3)$

$= (x-2)(x-3)(x-\sqrt{3})(x+\sqrt{3})$

<u>zeros</u> $\left.\begin{array}{l} x=2 \\ x=3 \\ x=\sqrt{3} \\ x=-\sqrt{3} \end{array}\right\}$ multiplicity 1, so f will change sign.

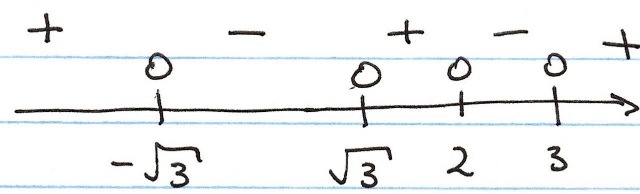
<u>End behavior</u>

If $|x|$ large, $f(x) \approx x^4$



so $\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = \infty$.



So $f(x) \geq 0$ when $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, 2] \cup [3, \infty)$