Questions

Sketch the following polynomials, by determining:

- x-intercepts with multiplicity
- vertical asymptotes with multiplicity
- end behaviour (show the end behaviour by determining $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Identify equations of any slant asymptotes using polynomial long division.

Also be on the lookout for holes in the graph!

1. $f(x) = \dfrac{(2x-1)^3(7-x)}{x(10-3x)^2}$.

2. $f(x) = \dfrac{9x^2-12x+4}{16-x^2}$.

3. $f(x) = \dfrac{(4x^2-1)^3}{x+1}$.

4. $f(x) = \dfrac{1}{3x-1} + \dfrac{1}{(3x-1)^2} + \dfrac{1}{x}$.

5. $f(x) = \dfrac{8x^3+1}{2x+1}$.

6. $f(x) = \dfrac{(3x^3-2x^2-3x+2)(x-1)}{x+4}$.

7. $f(x) = \dfrac{-4x^3-28x^2+9x+63}{-x^3-6x^2+9x+14}$.

Solve the following rational inequalities using a sign chart.

8. $\dfrac{(x-7)^4(x-15)^{13}}{x-199} \geq 0$.

9. $\dfrac{1}{x-5} \leq \dfrac{1}{x-3}$.

10. $\dfrac{x+4}{x+9} \leq -1$.

(1) $f(x) = \dfrac{(2x-1)^3(7-x)}{x\,(10-3x)^2}$

<u>zeros</u>: $x = 1/2$ multiplicity 3, so f changes sign.
Since muliplicity is greater than 1, f will be horizontal at the zero.

$x = 7$ multiplicity 1, so f changes sign.

<u>vertical asymptotes</u>

$x = 0$ multiplicity 1 so f changes sign

$x = 3/10$ multiplicity 2 so f does not change sign.

<u>End behaviour</u>

If $|x|$ is large, $f(x) \sim \dfrac{(2x)^3(-x)}{x\,(-3x)^2} = \dfrac{-8x^4}{9x^3} = -\dfrac{8}{9}x.$

This indicates a slant asymptote! To find equation of slant asymptote we have to multiply out and do polynomial long division.

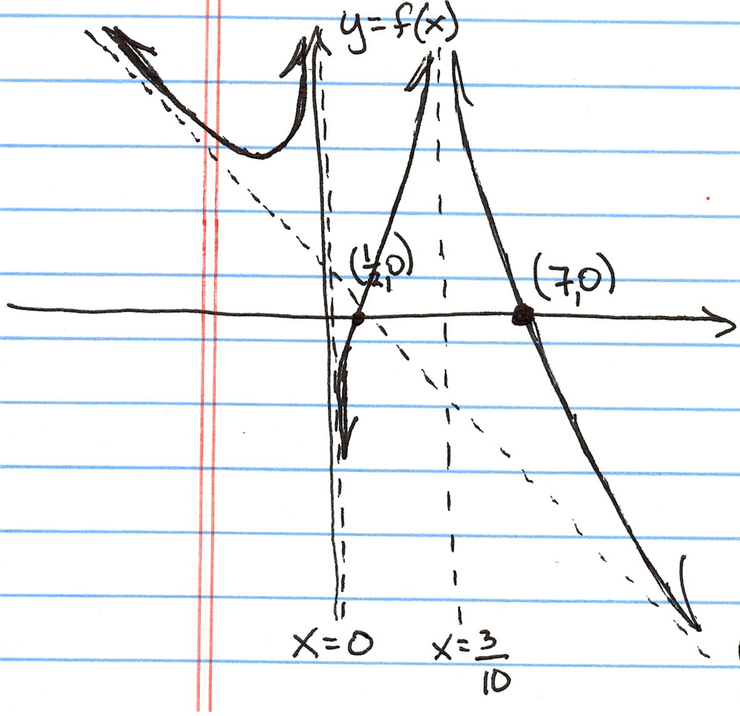
$$
\begin{aligned}
(2x-1)^3 &= (2x-1)^2(2x-1) \\
&= (4x^2-4x+1)(2x-1) \\
&= 8x^3-8x^2+2x-4x^2+4x-1 \\
&= 8x^3-12x^2+6x-1
\end{aligned}
$$

$$
\begin{aligned}
(2x-1)^3(7-x) &= (8x^3-12x^2+6x-1)(7-x) \\
&= 56x^3-84x^2+42x-7-8x^4+12x^3-6x^2+x \\
&= -8x^4+68x^3-100x^2+43x-7 \quad \text{(numerator)}
\end{aligned}
$$

$$x(10-3x)^2 = x(100-60x+9x^2)$$
$$= 100x - 60x^2 + 9x^3$$
$$= 9x^3 - 60x^2 + 100x \quad \text{(denominator)}$$

$$
\begin{array}{r|l}
 & -\frac{8}{9}x + \frac{44}{27} \\
\hline
9x^3 - 60x^2 + 100x & -8x^4 + 68x^3 - 100x^2 + 43x - 7 \\
 & -8x^4 + \frac{160}{3}x^3 - \frac{800}{9}x^2 \qquad \text{subtract.} \\
\hline
 & \frac{44}{3}x^3 - \frac{100}{9}x^2 + 43x \\
 & \frac{44}{3}x^3 - \frac{880}{9}x^2 + \frac{4400}{27}x \qquad \text{subtract} \\
\hline
 & \text{(we can stop).}
\end{array}
$$

All this work was simply to get the y-intercept of the slant asymptote (notice we already knew slope was $-8/9$).
slant asymptote is $y = -\frac{8}{9}x + \frac{44}{27}$.



$y = -\frac{8}{9}x + \frac{44}{27}$ (slant asymptote)

From sketch, we can see

$\lim_{x\to \frac{10}{3}^+} f(x) = \infty$ $\qquad \lim_{x\to\infty} f(x) = -\infty$

$\lim_{x\to \frac{10}{3}^-} f(x) = \infty$ $\qquad \lim_{x\to-\infty} f(x) = \infty$

$\lim_{x\to 0^+} f(x) = -\infty$

$\lim_{x\to 0^-} f(x) = +\infty$

Domain $x \in (-\infty,0)\cup(0,\frac{10}{3})\cup(\frac{10}{3},\infty)$.

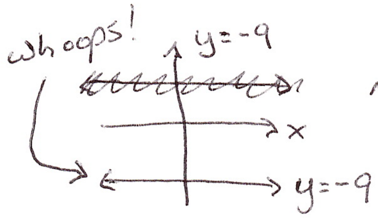
2) Sketch $f(x) = \dfrac{9x^2-12x+4}{16-x^2}$

Factor $9x^2-12x+4 = (3x-2)^2$ perfect square

$16-x^2 = (4-x)(4+x)$ difference of squares

$$f(x) = \frac{(3x-2)^2}{(4-x)(4+x)}$$

<u>End Behaviour:</u> If $|x|$ is large, $f(x) \sim \dfrac{9x^2}{-x^2} = -9$. Therefore there is a horizontal asymptote of $y=-9$.

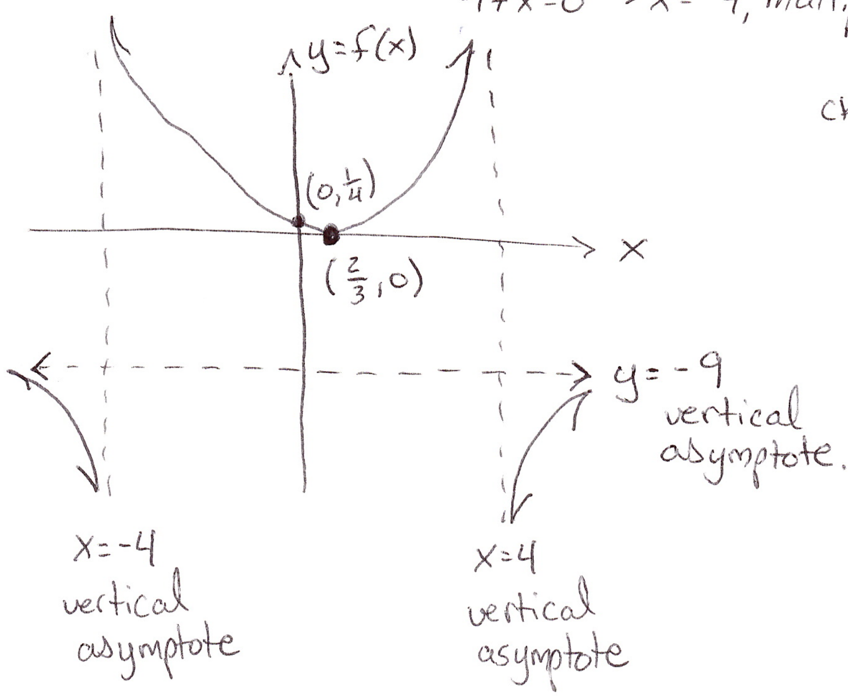


Note: $\lim\limits_{x\to\pm\infty} f(x) = -9$

<u>Zeros:</u> $3x-2=0 \to x=\frac{2}{3}$, multiplicity 2 (even) so f does not change sign.

<u>vertical asymptotes:</u> $4-x=0 \to x=4$, multiplicity 1 (odd) so f changes sign.

$4+x=0 \to x=-4$, multiplicity 1 (odd) so f changes sign.



Check: $f(0) = \frac{4}{16} = \frac{1}{4}$

so y intercept is $\frac{1}{4}$, which agrees with our sketch.

From the sketch we see that

$\lim\limits_{x\to 4^+} f(x) = -\infty$

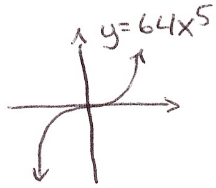
$\lim\limits_{x\to 4^-} f(x) = \infty$

$\lim\limits_{x\to -4^+} f(x) = \infty$

$\lim\limits_{x\to -4^-} f(x) = -\infty$

3) Sketch $f(x) = \dfrac{(4x^2-1)^3}{x+1}$

<u>End behaviour</u>: If $|x|$ is large, $f(x) \sim \dfrac{(4x^2)^3}{x} = \dfrac{64x^6}{x} = 64x^5$.



Note: $\lim_{x \to \infty} f(x) = \infty$ $\lim_{x \to -\infty} f(x) = -\infty$. No horizontal or slant asymptotes.

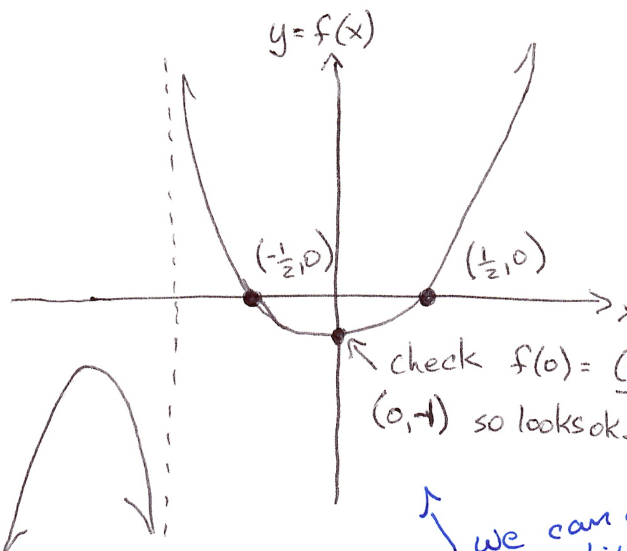
we need to factor before we find zeros.

$$4x^2-1 = (2x)^2 - 1^2 = (2x-1)(2x+1) \text{ difference of squares.}$$

$$f(x) = \frac{(2x-1)^3(2x+1)^3}{x+1}$$

<u>Zeros</u>: $2x-1=0 \to x=\frac{1}{2}$, multiplicity 3 (odd) so f changes sign.
$2x+1=0 \to x=-\frac{1}{2}$, multiplicity 3 (odd) so f changes sign.

<u>vertical asymptotes</u>: $x+1=0 \to x=-1$, multiplicity 1 (odd) so f changes sign.



check $f(0) = \frac{(-1)^3}{1} = -1$
$(0,-1)$ so looks ok.

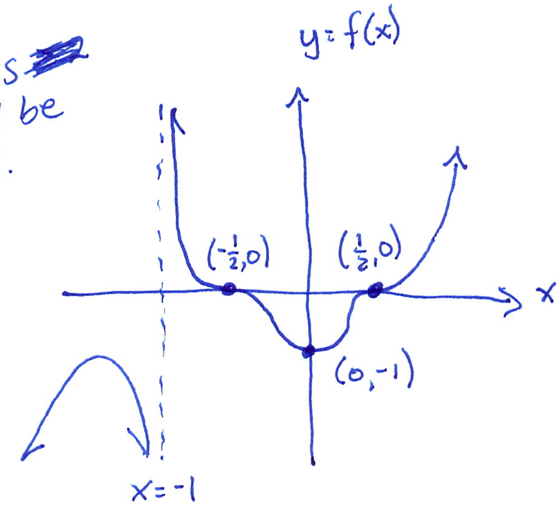
$x=-1$
vertical asymptote

From the sketch we can see that:

$$\lim_{x \to -1^+} f(x) = \infty$$

$$\lim_{x \to -1^-} f(x) = -\infty$$

we can do better than this! Since multiplicity of zeros is $3 \geq 2$, the function will be horizontal at the zeros.



$x=-1$

4) Sketch $f(x) = \frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{x}$

First get a common denominator:

$$f(x) = \frac{1}{3x-1} \cdot \frac{(3x-1)x}{(3x-1)x} + \frac{1}{(3x-1)^2} \cdot \frac{x}{x} + \frac{1}{x} \cdot \frac{(3x-1)^2}{(3x-1)^2}$$

$$= \frac{(3x-1)x + x + (3x-1)^2}{(3x-1)^2 x}$$

$$= \frac{3x^2 - x + x + 9x^2 - 6x + 1}{(3x-1)^2 x}$$

$$= \frac{12x^2 - 6x + 1}{(3x-1)^2 x}$$

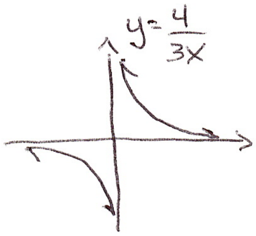
Factor $12x^2 - 6x + 1$: use quadratic formula:

$$x = \frac{6 \pm \sqrt{36 - 4(12)(1)}}{2 \cdot 12}$$

$$= \frac{6 \pm \sqrt{-12}}{24}$$

no real roots, so quadratic is irreducible.

<u>End behaviour</u>: If $|x|$ is large, $f(x) \sim \frac{12x^2}{(3x)^2 x} = \frac{12x^2}{9x^3} = \frac{4}{3x}$



Note: $\lim_{x \to \infty} f(x) = 0$

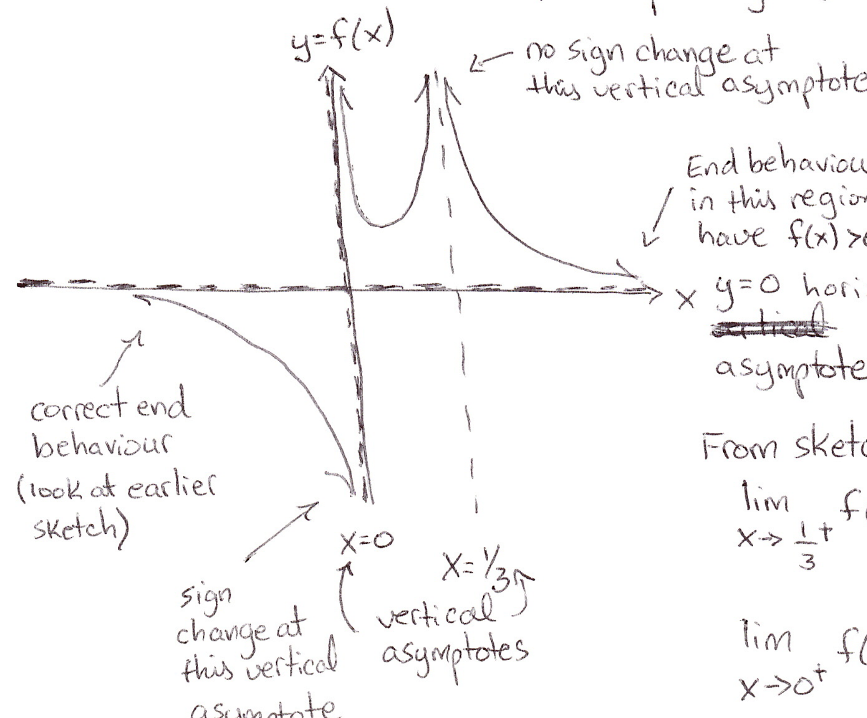
$\lim_{x \to -\infty} f(x) = 0$

So f has a horizontal asymptote of $y = 0$.

<u>Zeros</u>: no zeros since $12x^2 - 6x + 1$ is irreducible.

<u>vertical asymptotes</u>: $(3x-1) = 0 \to x = \frac{1}{3}$, multiplicity 2 (even) so f does not change sign.

$x = 0$, multiplicity 1 (odd) so f changes sign.



From sketch:

$\lim_{x \to \frac{1}{3}^+} f(x) = \infty$ $\quad \lim_{x \to \frac{1}{3}^-} f(x) = \infty$ $\quad$ since they both approach same limit, we can combine these and say $\lim_{x \to \frac{1}{3}} f(x) = \infty$

$\lim_{x \to 0^+} f(x) = \infty$ $\quad \lim_{x \to 0^-} f(x) = -\infty$

5 Sketch $f(x) = \dfrac{8x^3+1}{2x+1}$

Factor $8x^3+1$. Sum of cubes. use formula, or notice $8x^3+1\Big|_{x=-1/2} = 0 \Rightarrow x+\frac{1}{2}$ is a factor.

$$\begin{array}{r l} & 8x^2-4x+2 \\ x+\frac{1}{2} \,\big) & 8x^3+0x^2+0x+1 \\ & \underline{8x^3+4x^2} \quad \text{subtract} \\ & -4x^2+0x \\ & \underline{-4x^2-2x} \\ & 2x+1 \\ & \underline{2x+1} \\ & 0 \end{array}$$

Note: $8x^2-4x+2$ is irreducible since

$$x = \frac{4 \pm \sqrt{16-4(8)(2)}}{16} = \frac{4 \pm \sqrt{-48}}{16} \text{ not a real number.}$$

$$f(x) = \frac{(x+\frac{1}{2})(8x^2-4x+2)}{2x+1}$$

Note: numerator and denominator are both zero when $x = -1/2$, which means we have a hole at $x=-1/2$. Rewrite to cancel the factor.

$$= \frac{(x+\frac{1}{2})\,2\,(4x^2-2x+1)}{2x+1}$$

$$= \frac{\cancel{(2x+1)}(4x^2-2x+1)}{\cancel{(2x+1)}}$$

$= 4x^2-2x+1$, $\quad 2x+1 \neq 0 \rightarrow x \neq -1/2$. This is a quadratic

- opens up since $a=4>0$.
- no zeros since $4x^2-2x+1$ is irreducible
- get vertex by completing the square.
- hole at $x=-1/2$

$$y = f(-\tfrac{1}{2}) = 4(-\tfrac{1}{2})^2 - 2(-\tfrac{1}{2}) + 1 = 3$$

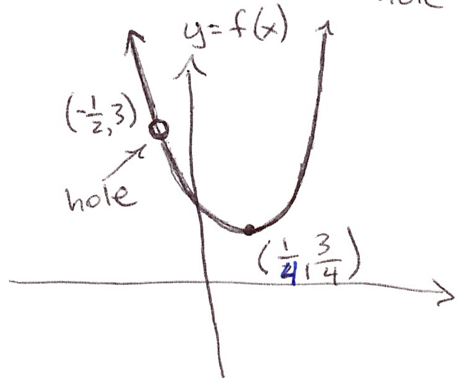
hole at $(-\frac{1}{2}, 3)$.

$$4x^2-2x+1 = 4\left(x^2 - \tfrac{1}{2}x + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2\right) + 1$$
$$= 4\left([x-\tfrac{1}{4}]^2 - \tfrac{1}{16}\right) + 1$$
$$= 4[x-\tfrac{1}{4}]^2 - \tfrac{4}{16} + 1$$
$$= 4[x-\tfrac{1}{4}]^2 + \tfrac{3}{4}$$

So $f(x) = 4[x-\frac{1}{4}]^2 + \frac{3}{4}$, $x \neq -1/2$

vertex form $y = a(x-h)^2+k$.

$\Rightarrow (h,k) = (\frac{1}{4}, \frac{3}{4})$



6 Sketch $f(x) = \dfrac{(3x^3-2x^2-3x+2)(x-1)}{x+4}$

Factor $3x^3-2x^2-3x+2$

Factors of $a_0=+2$: $\pm1, \pm2$
Factors of $a_3=3$: $\pm1, \pm3$

Possible rational factors: $\pm1, \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}$.

$3(1)^3-2(1)^2-3(1)-2=0$, so $x-1$ will factor.

$$\begin{array}{r l}
 & 3x^2+x-2 \\
x-1 \,\big) & 3x^3-2x^2-3x+2 \\
 & \underline{3x^3-3x^2} \quad \text{subtract} \\
 & x^2-3x \\
 & \underline{x^2-x} \\
 & -2x+2 \\
 & \underline{-2x+2} \\
 & 0
\end{array}$$

so $3x^3-2x^2-3x+2 = (x-1)(3x^2+x-2)$
$= (x-1)(3x-2)(x+1)$

If you need to factor $3x^2+x-2$ using quadratic formula, here are the details:

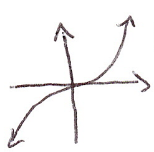
$x = \dfrac{-1 \pm \sqrt{1-4(3)(-2)}}{6} = \dfrac{-1\pm5}{6} = -1$ or $\dfrac{2}{3}$

factor $(x+1)$ factor $x-\frac{2}{3}$

So $3x^2+x-2 = 3(x+1)(x-\frac{2}{3})$
$= (x+1)(3x-2)$.

So $f(x) = \dfrac{(x-1)(3x-2)(x+1)(x-1)}{x+4}$
$= \dfrac{(3x-2)(x+1)(x-1)^2}{x+4}$ Collect $(x-1)$ factors together.

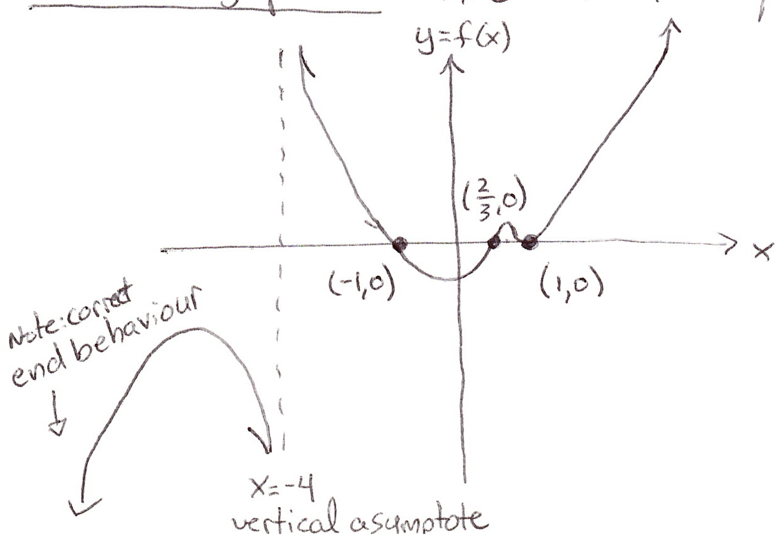
End behaviour: If $|x|$ is large, $f(x) \sim \dfrac{(3x)(x)(x)^2}{x} = 3x^3$



so $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$.

Zeros: $3x-2=0 \to x=2/3$
$x+1=0 \to x=-1$ } multiplicity 1 (odd) so f changes sign.
$x-1=0 \to x=1$ multiplicity 2 (even) so f does not change sign.

vertical asymptotes: $x+4=0 \to x=-4$ multiplicity 1 (odd) so f changes sign.



From the sketch we can see that

$\lim\limits_{x\to-4^+} f(x) = \infty$

$\lim\limits_{x\to-4^-} f(x) = -\infty$

7

Analyze $f(x) = \dfrac{-4x^3 - 28x^2 + 9x + 63}{-x^3 - 6x^2 + 9x + 14}$.

Factor. Use Rational Zero Theorem to look for rational factors.

numerator: $a_0 = 63: \pm 1, \pm 7, \pm 9, \pm 63$

$a_3 = -4: \pm 1, \pm 2, \pm 4$

Possible rational factors:
$\pm 1, \pm 7, \pm 9, \pm 63, \pm \frac{1}{2}, \pm \frac{7}{2}, \pm \frac{9}{2}, \pm \frac{63}{2}, \pm \frac{1}{4}, \pm \frac{7}{4}, \pm \frac{9}{4}, \pm \frac{63}{4}$.

Notice $-4x^3 - 28x^2 + 9x + 63 \Big|_{x=-7} = 0 \Rightarrow x + 7$ is a factor.

$$\begin{array}{r|l} & -4x^2 + 9 \\ \hline x+7 & -4x^3 - 28x^2 + 9x + 63 \\ & -4x^3 - 28x^2 \\ \hline & \qquad\qquad 9x + 63 \\ & \qquad\qquad 9x + 63 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

$\Rightarrow -4x^3 - 28x^2 + 9x + 63 = (x+7)(-4x^2 + 9)$

$= (x+7)(9 - 4x^2)$ difference of squares

$= (x+7)(3-2x)(3+2x)$

denominator: $a_0 = 14: \pm 1, \pm 2, \pm 7, \pm 14$

$a_3 = -1: \pm 1$

Possible rational factors:
$\pm 1, \pm 2, \pm 7, \pm 14$

Notice $-x^3 - 6x^2 + 9x + 14 \Big|_{x=-1} = 0 \Rightarrow x + 1$ is a factor.

$$\begin{array}{r|l} & -x^2 - 5x + 14 \\ \hline x+1 & -x^3 - 6x^2 + 9x + 14 \\ & -x^3 - x^2 \\ \hline & \quad -5x^2 + 9x \\ & \quad -5x^2 - 5x \\ \hline & \qquad\quad 14x + 14 \\ & \qquad\quad 14x + 14 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow -x^3 - 6x^2 + 9x + 14 = (x+1)(-x^2 - 5x + 14)$

$= -(x+1)(x^2 + 5x - 14)$

$= -(x+1)(x-2)(x+7)$

So $f(x) = \dfrac{\cancel{(x+7)}(3-2x)(3+2x)}{-(x+1)(x-2)\cancel{(x+7)}}$

$$\boxed{f(x) = \frac{-(3-2x)(3+2x)}{(x+1)(x-2)}, \quad x \neq -7}$$

Hole $(-7, f(-7)) = (-7, \frac{187}{54})$

Zeros $x = 3/2, -3/2$ both multiplicity 1.

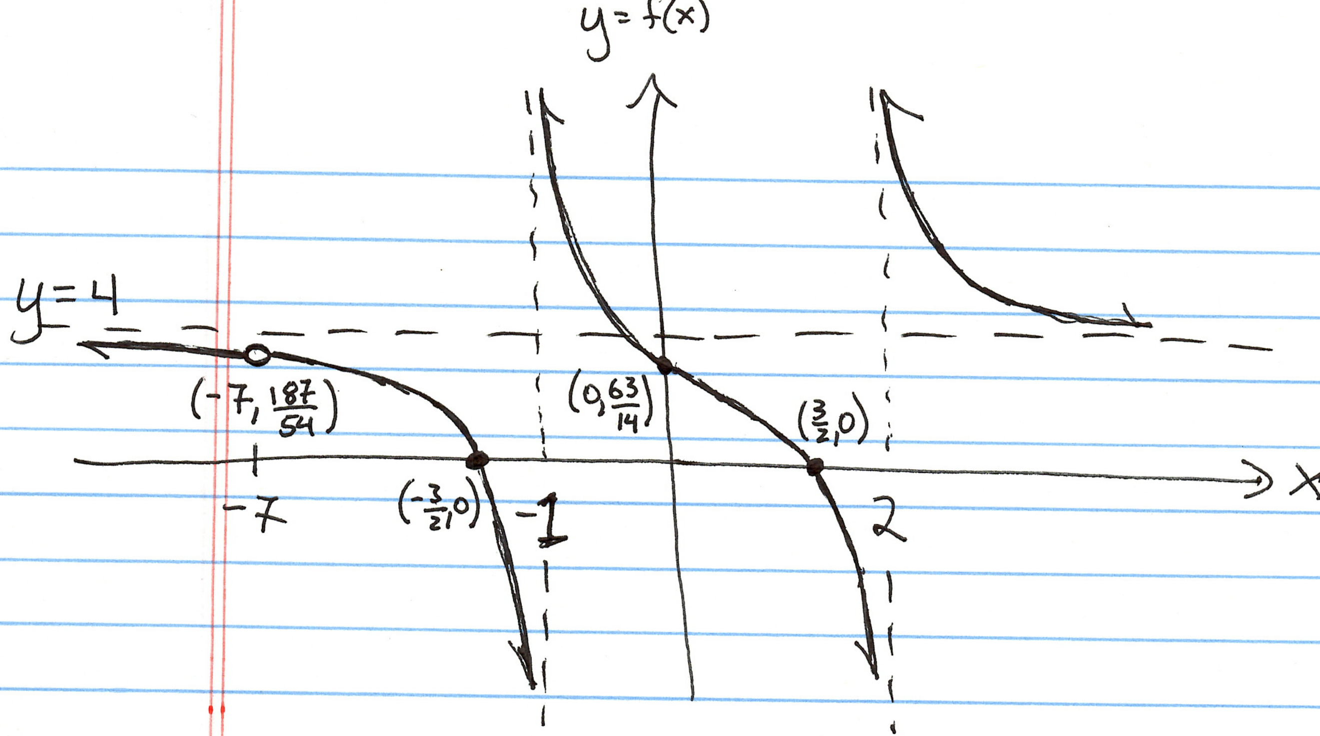
Vertical asymptotes $x = -1, 2$ both multiplicity 1.

End behaviour If $|x|$ large, $f(x) \sim \dfrac{-(-2x)(2x)}{(x)(x)} = 4$.

There is a horizontal asymptote of $y = 4$.

Domain will exclude any holes or vertical asymptotes. Domain is

$x \in (-\infty, -7) \cup (-7, -1) \cup (-1, 2) \cup (2, \infty)$



y = f(x)
y = 4
(-7, 187/54)
(0, 63/14)
(3/2, 0)
(-3/2, 0)
-7
-1
2
x

(8) $f(x) = \dfrac{(x-7)^4 (x-15)^{13}}{x-199}$

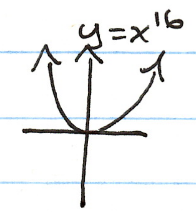
zeros: $x=7$ multiplicity 4, so f will not change sign.
$x=15$ multiplicity 13, so f will change sign.

vertical asymptotes

$x=199$, multiplicity 1, so f changes sign.

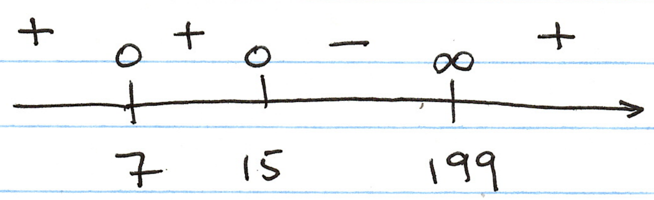
End behaviour

If $|x|$ is large, $f(x) \sim \dfrac{(x)^4 (x)^{13}}{x} = x^{16}$



so $\lim\limits_{x\to\infty} f(x) = \infty$

$\lim\limits_{x\to -\infty} f(x) = \infty$



$f(x) \geq 0$ if $x \in (-\infty, 15] \cup (199, \infty)$

(9) Rewrite as $f(x) \leq 0$

$$\frac{1}{x-5} \leq \frac{1}{x-3}$$

$$\frac{1}{x-5} - \frac{1}{x-3} \leq 0$$

$$\frac{x-3-x+5}{(x-5)(x-3)} \leq 0$$

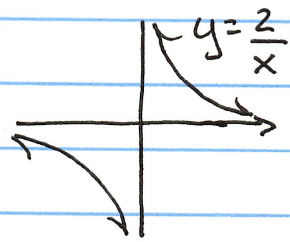
$$f(x) = \frac{2}{(x-5)(x-3)} \leq 0$$

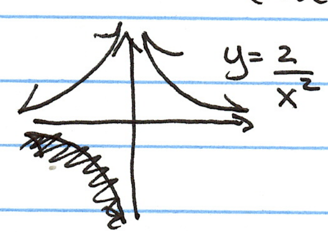
<u>zeros</u> none

<u>vertical asymptotes</u> $\left.\begin{array}{l} x=5 \\ x=3 \end{array}\right\}$ multiplicity 1, so f changes sign.

<u>End behaviour</u>

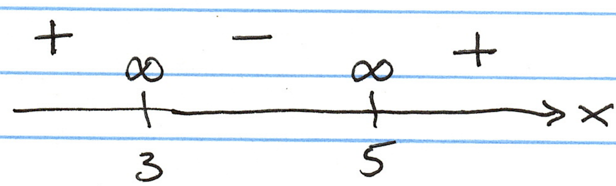
If $|x|$ is large, $f(x) \sim \frac{2}{(x)(x)} = \frac{2}{x^2}$





$\lim\limits_{x\to\infty} f(x) = 0$ (positive)

$\lim\limits_{x\to-\infty} f(x) = 0$ (positive)



so $f(x) \leq 0$ if $x \in (3,5)$.

(10) $\dfrac{x+4}{x+9} < -1$ Rewrite as $f(x) < 0$

$$\frac{x+4}{x+9} + 1 < 0$$

$$\frac{x+4+x+9}{x+9} < 0$$

$$f(x) = \frac{2x+13}{x+9} < 0$$

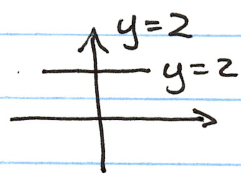
<u>zeros</u> $x = -13/2$ multiplicity 1 so f changes sign.

<u>vertical asymptotes</u> $x = -9$ multiplicity 1 so f changes sign.

<u>End behaviour</u>

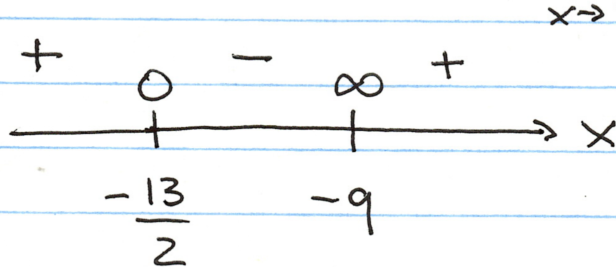
If $|x|$ is large $f(x) \sim \dfrac{2x}{x} = 2$ (positive)

So



$\lim_{x \to \infty} f(x) = 2$ (positive)

$\lim_{x \to -\infty} f(x) = 2$ (positive)



$f(x) < 0$ when $x \in \left(-\frac{13}{2}, -9\right)$