Solving Equations

1. Solve $\ln(x) + \ln(x+2) = \ln 8$.

2. Solve $\ln(x+1) + \ln(x-1) = 2$.

3. Solve $\ln\left(\frac{x-3}{2}\right) + \ln\left(\frac{x+2}{7}\right) = 1$.

4. Solve the equation $\frac{2^x - 2^{-x}}{3} = 4$ algebraically for x.

5. Solve the equation $\frac{1}{2}\ln(x+3) = \ln x$.

6. Solve the equation $\ln(t-2) + \ln(t+5) = 2\ln 3$.

Modeling When Model Function is Given

7. The time it takes to payoff a loan is given by the formula

$$R = P\frac{i}{1-(1+i)^{-nt}}$$

where R is monthly payment, P is amount borrowed, n is the number of payments per year, t is the number of year, r is the annual interest rate per year, and $i = r/n$. Solve for t.

8. Newton's law of cooling is

$$D(t) = D_0 e^{kt}$$

where D_0 is the initial difference in temperature between object and room temperature. $D(t)$ is the difference in temperature at time t, t is time and k is constant that is determined by the situation.

Marlene brought a can of varnish stored at 40F into her shop, where the temperature was 74F. After 2hr the temperature of the varnish was 58F. If the varnish must be at 68F for best results, how much longer must Marene wait to use the varnish?

9. Biologists use the species-are curve $n = k\log_{10}(A)$ to estimate the number of species n that live in a region of area A, where k is a constant.

(a) If 2500 species live in a rainforest of 400 km^2, how many species would be left if half the rainforest was removed by logging?

(b) A rainforest of 1200 km^2 supported 3500 species in 1950. In 2000, the rainforest only supported 1000 species in 2000. What percentage of the rainforest has been destroyed from 1950 to 2000?

10. Logistic growth models for how a virus moves through populations are

$$n = \frac{P}{1+(P-1)e^{-ct}}$$

where P is population size, n is the number of people who have caught the virus before day t, and c is a positive constant determined by data.

If $P = 10{,}000$, $c = 0.2$, determine the time when half the population has caught the virus.

Building Your Own Model Function

11. The half life of a certain radioactive material is 65 days. There are initially 4 grams of the material present. Find an expression for the amount of material t **days** after the initial measurement of 4 grams. How many days will it take for there to be only 1 gram left? (give the answer both exactly and as a decimal).

Show all you calculations. Derive any formulas you need, do not simply plug numbers into a population growth formula you have memorized.

12. A population of 200 fish is released into a lake. The population doubling time for this breed of fish is 15 months. If the plan is to allow limited fishing on the lake once the fish population exceeds 10000 fish, when should fishing be allowed to begin (give the answer both exactly and as a decimal)?

Show all you calculations. Derive any formulas you need, do not simply plug numbers into a population growth formula you have memorized.

(1) $\ln(x) + \ln(x+2) = \ln 8$

$\left.\begin{array}{l} x > 0 \\ x+2 > 0 \end{array}\right\}$ both must be true $\Rightarrow x > 0.$

$\ln(x(x+2)) = \ln 8$

$e^{\ln(x(x+2))} = e^{\ln 8}$

$x(x+2) = 8$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4, \quad x = 2.$

$\uparrow$
extraneous.

one solution, $x = 2$.

(2) $\ln(x+1) + \ln(x-1) = 2$

$\left.\begin{array}{l} x+1 > 0 \\ x-1 > 0 \end{array}\right\} \Rightarrow x > 1.$

$\ln((x+1)(x-1)) = 2$

$\ln(x^2-1) = 2$

$e^{\ln(x^2-1)} = e^2$

$x^2 - 1 = e^2$

$x = \pm\sqrt{e^2+1}$

$x = -\sqrt{e^2+1}$ extraneous.

$x = \sqrt{e^2+1} > 1$ only solution.

(3) $\ln\left(\frac{x-3}{2}\right) + \ln\left(\frac{x+2}{7}\right) = 1$

$\frac{x-3}{2} > 0$ and $\frac{x+2}{7} > 0$ $\Rightarrow$ $x > 3$ and $x > -2$ $\Rightarrow$ $x > 3$

$$\ln\left(\left(\frac{x-3}{2}\right)\left(\frac{x+2}{7}\right)\right) = 1$$

$$\ln\left(\frac{x^2-x-6}{14}\right) = 1$$

$$e^{\ln\left(\frac{x^2-x-6}{14}\right)} = e^1$$

$$\frac{x^2-x-6}{14} = e$$

$$x^2 - x - 6 = 14e$$

$$x^2 - x + (-6-14e) = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1 - 4(1)(-6-14e)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{25+56e}}{2}$$

$x = \frac{1-\sqrt{25+56e}}{2} < 3$ is extraneous.

$x = \frac{1+\sqrt{25+56e}}{2} > 3$ is only solution.

(4) $2^x - 2^{-x} = 12$ Multiply by 2^x to remove 2^{-x}.

$(2^x)^2 - 1 = 12 \cdot 2^x$

$(2^x)^2 - 12(2^x) - 1 = 0$ u-substitution. $u = 2^x$

$u^2 - 12u - 1 = 0$

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{144 - 4(1)(-1)}}{2(1)}$$

$$= \frac{12 \pm \sqrt{148}}{2}$$

$u = 2^x = \frac{12 + \sqrt{148}}{2}$ or $u = 2^x = \frac{12 - \sqrt{148}}{2}$

$x = \log_2\left(\frac{12 + \sqrt{148}}{2}\right)$ $\quad$ $x = \log_2\left(\frac{12 - \sqrt{148}}{2}\right)$

one solution. $\quad$ extraneous, since $\frac{12 - \sqrt{148}}{2} < 0$.

(5) $\frac{1}{2}\ln(x+3) = \ln(x)$

$\ln(\sqrt{x+3}) = \ln(x)$

$e^{\ln(\sqrt{x+3})} = e^{\ln(x)}$

$(\sqrt{x+3})^2 = (x)^2$

$x+3 = x^2$

$x^2 - x - 3 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{1 \pm \sqrt{1-4(1)(-3)}}{2(1)}$

$= \frac{1 \pm \sqrt{13}}{2}$

$x+3>0 \rightarrow x>0.$

$x>0$

$x = \frac{1+\sqrt{13}}{2}$ solution

~~check~~

$x = \frac{1-\sqrt{13}}{2} < 0$ extraneous

(6) $\ln(t-2) + \ln(t+5) = 2\ln(3)$

$\ln((t-2)(t+5)) = \ln(9)$

$(t-2)(t+5) = 9$

$t^2 + 3t - 10 = 9$

$t^2 + 3t - 19 = 0$

$t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-3 \pm \sqrt{9-4(1)(-19)}}{2(1)}$

$= \frac{-3 \pm \sqrt{85}}{2}$

$t-2>0 \rightarrow t>2.$

$t+5>0$

$t = \frac{-3-\sqrt{85}}{2} < 2$ extraneous

$t = \frac{-3+\sqrt{85}}{2} > 2$ only solution.

(7) $R = P\left(\dfrac{i}{1-(1+i)^{-nt}}\right)$

$1-(1+i)^{-nt} = \dfrac{Pi}{R}$

$1-\dfrac{Pi}{R} = (1+i)^{-nt}$

$\ln\left(1-\dfrac{Pi}{R}\right) = \ln\left((1+i)^{-nt}\right)$

$\ln\left(1-\dfrac{Pi}{R}\right) = -nt\ln(1+i)$

$t = \dfrac{-\ln(1-Pi/R)}{n\ln(1+i)}$

(8) $D(t) = D_0 e^{kt}$ $\quad$ $D(t)$ is difference in temp after t hours

$D_0 = 74-40 = 34$

$D(2) = 74-58 = 34e^{k(2)}$ solve for k.

$16 = 34e^{2k}$

$8/17 = e^{2k}$

$\ln(8/17) = \ln(e^{2k})$

$\ln(8/17) = 2k$

$\to k = \frac{1}{2}\ln(8/17)$

Model is

$D(t) = 34e^{\frac{1}{2}\ln(8/17)t}$

Temp will be 68 when

$D(t) = 74-68 = 6 = 34e^{\frac{1}{2}\ln(8/17)t}$

solve for t.

$3/17 = e^{\frac{1}{2}\ln(8/17)t}$

$\ln(3/17) = \ln e^{\frac{1}{2}\ln(8/17)t}$

$\ln(3/17) = \frac{1}{2}\ln(8/17)t$

$t = \dfrac{2\ln(3/17)}{\ln(8/17)}$ hours

~ 4.602 hours.

8) $n = k\log_{10}(A)$

a) Use data to get k.

$2500 = k\log_{10}(400) \rightarrow k = \dfrac{2500}{\log_{10}(400)}$

Model: $n = \dfrac{2500}{\log_{10}(400)} \log_{10}(A)$

$= \dfrac{2500}{\log_{10}(400)} \log_{10}(200)$ (half rainforest removed)

~ 2210 species.

b) Use data to get k:

$3500 = k\log_{10}(1200) \longrightarrow k = \dfrac{3500}{\log_{10}(1200)}$

Model: $n = \dfrac{3500}{\log_{10}(1200)} \log_{10}(A)$

Solve for A: $\dfrac{\log_{10}(1200)}{3500} n = \log_{10}(A)$

$\rightarrow A = 10^{\left(\frac{\log_{10}(1200)}{3500} n\right)}$ area

$A = 10^{\left(\frac{\log_{10}(1200)}{3500} \cdot 1000\right)}$

$\sim 7.58\ km^2$

Percentge left $= 7.58/1200 \sim 0.006317$

Percentage destroyed $= 1 - 0.006317 \sim 0.9937 \hat{=} 99\%$

(10) $n = \dfrac{P}{1+(P-1)e^{-ct}}$

$n=5000$ when half population is infected.

$$5000 = \frac{10000}{1+9999e^{-0.2t}}$$

$$1+9999e^{-0.2t} = 2$$

$$e^{-0.2t} = \frac{1}{9999}$$

$$-0.2t = \ln\left(\frac{1}{9999}\right)$$

$$t = -\frac{1}{0.2}\ln\left(\frac{1}{9999}\right) \sim 46 \text{ days.}$$

(11) Get Pattern from given information.

$m(0) = 4$ grams

t is in days.

$m(65) = 2$ grams $= \frac{1}{2}\cdot 4$

$m(2\cdot 65) = \left(\frac{1}{2}\right)^2\cdot 4$

$m(3\cdot 65) = \left(\frac{1}{2}\right)^3\cdot 4$

$\vdots$ Look for pattern

$$m(t) = \left(\frac{1}{2}\right)^{t/65}\cdot 4 = 4\left(\frac{1}{2}\right)^{t/65}$$

$$= 4\cdot 2^{-t/65} \text{ is mass after } t \text{ days.}$$

There will be 1 gram left when $1 = 4\cdot 2^{-t/65}$

$$\frac{1}{4} = 2^{-t/65}$$

$$\ln(1/4) = \ln\left(2^{-t/65}\right)$$

$$\ln(1/4) = -\frac{t}{65}\ln(2)$$

$$t = \frac{-65\ln(1/4)}{\ln(2)}$$

$$= 130 \text{ days}$$

Note: $t = \dfrac{-65\ln(1/4)}{\ln(2)} = \dfrac{-65\ln(2^{-2})}{\ln(2)}$

$$= -65(-2)\frac{\cancel{\ln(2)}}{\cancel{\ln(2)}} = 130!$$

(12) Let t be in months.

$P(0) = 200$ fish

$P(15) = 2 \cdot 200$

$P(2 \cdot 15) = 2^2 \cdot 200$

$P(3 \cdot 15) = 2^3 \cdot 200$

$P(4 \cdot 15) = 2^4 \cdot 200$

$\vdots$ look for pattern

$$P(t) = 2^{t/15} \cdot 200$$

There will be 10 000 fish when

$$10\,000 = 2^{t/15} \cdot 200 \qquad \text{solve for } t.$$

$$50 = 2^{t/15}$$

$$\ln(50) = \ln(2^{t/15})$$

$$\ln(50) = \frac{t}{15} \ln(2)$$

$$t = \frac{15 \ln(50)}{\ln(2)} \text{ months}$$

$$\approx 84.6 \text{ months}.$$

Fishing may begin 85 months after the fish are released.