Math 1012 Precalculus I Functions Test 3
Practice

There will be no Descartes's Rule of Signs on the Test.

Discussion

Make some notes on the questions, then discuss what you wrote with a partner. Make note of anything that you and your partner disagree on, or that you had difficulty with.

1. Write down in words the things you need to sketch a polynomial.

2. Write down in words the things you need to sketch a rational function.

3. What are the three types of asymptotes? Draw example sketches of each, and include correct limit notation to explain each asymptote.

4. What do you use a sign chart to do? Why?

5. Explain in words how you solve the following types of equations:

- Equations involving square roots
- Equations of quadratic type
- Equations involving absolute values

6. Explain the technique in words to factor $f(x) = 12x^3 - 16x^2 + 7x - 1$.

7. Explain what an extraneous solution is.

8. Explain what a hole is, in relation to a rational function.

9. How can you tell if a function changes sign at a zero or vertical asymptote?

Computation

Work out solutions, and discuss with a partner as needed. Make sure your solutions are well organized, complete, and use correct mathematical notation. Make sure <u>any</u> sketch you draw is labelled.

10. Solve $\sqrt{x^2+1} + \sqrt{x^2-2} = 7$.

11. Sketch $f(x) = \dfrac{(3x-1)^2(x+1)}{(x-1)^3}$.

12. Solve $|2x - x^2| = 2x - 4$.

13. Solve $|2x| = 2 - |x|$.

14. Find all real and complex valued solutions to $12x^4 - 16x^3 + 36x^2 - 48x = 0$. ~~the largest poten~~

15. Determine, in interval notation, the solution to $x \leq \dfrac{1}{(x-2)^2}$.

16. Sketch $g(x) = \dfrac{(3x-1)^2 x}{(x-1)^2}$.

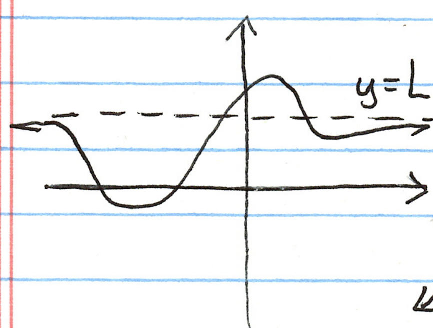
17. Sketch $f(x) = 7(2 - x)^3(x^2 - 2x + 1)$.

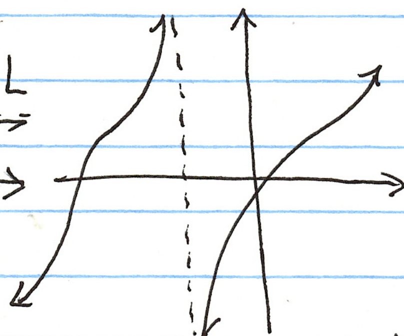
18. Solve $(x^2 + 2x)^2 - (x^2 + 2x) - 2 = 0$.

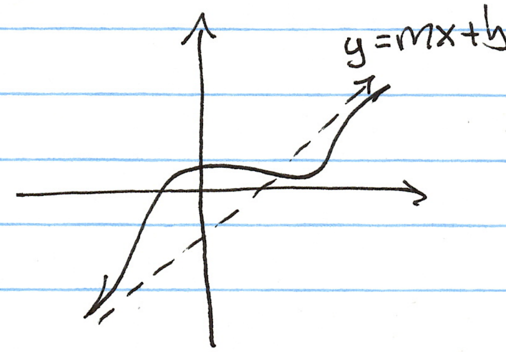
(1) zeros with multiplicity
end behaviour

(2) zeros with multiplicity
vertical asymptotes with multiplicity
end behaviour $\longrightarrow$ watch for horizontal & slant asymptotes
Watch for holes!

(3)







horizontal	$x=a$ vertical	slant
$\lim\limits_{x\to\infty}\left[f(x)\right]=L$	$\lim\limits_{x\to a^+}\left[f(x)\right]=-\infty$	$\lim\limits_{x\to\infty}\left[f(x)\right]=mx+b$
$\lim\limits_{x\to-\infty}\left[f(x)\right]=L$	$\lim\limits_{x\to a^-}\left[f(x)\right]=\infty$	$\lim\limits_{x\to-\infty}\left[f(x)\right]=mx+b$
(f can cross)	(f cannot cross)	

(4) Sign charts can be solve inequalities. They are necessary since inequalities are difficult to solve algebraically.
~~when you~~

(5) square roots: isolate a square root, then square both sides of equation.

Note $(A+B)^2 = A^2+B^2+2AB$

$\sqrt{A+B} \neq \sqrt{A}+\sqrt{B}$

Quadratic Type: Do a substitution, u = the piece that appears twice.

Solve for u.

Put values for u into and solve for x.

Absolute Values: Use $|f(x)| = k > 0$ is equivalent to $f(x) = k$ or $f(x) = -k$.

In all cases: eliminate extraneous solutions.

Be prepared to combine these techniques.

(6) Get factors of -1

Get factors of 12

Take Ratio — this gives you potential rational factors.

Find factors. $f(c)=0 \longrightarrow x-c$ is a factor of $f(x)$.

When you get to quadratic, use quadratic formula

$$ax^2+bx+c=0 \longrightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

(7) Algebraic manipulations introduce a potential solution that is not a solution of the original equation.

(8) A hole occurs when the same ~~x value~~ $x-c$ is a factor of the numerator and denominator in a rational expression.

(9) If the zero or vertical asymptote has odd multiplicity the function will change sign. Even multiplicity there will not be a sign change.

(10) $\left(\sqrt{x^2+1}\right)^2 = \left(7-\sqrt{x^2-2}\right)^2$ AHA! Caught an error!

$x^2+1 = 49 - 14\sqrt{x^2-2} + (x^2-2)$

$\cancel{2x^2-52 = 14\sqrt{x^2-2}}$

$\cancel{(x^2-25)^2 = \left(7\sqrt{x^2-2}\right)^2}$

$\cancel{x^4-50x^2+625 = 49(x^2-2)}$

$\cancel{x^4-99x^2+723=0}$

$-46 = -14\sqrt{x^2-2}$

$\left(\frac{23}{7}\right)^2 = \left(\sqrt{x^2-2}\right)^2$

$\frac{529}{49} = x^2-2$

$x^2 = \frac{627}{49}$

$x = \pm\frac{\sqrt{627}}{7}$

check:

$\sqrt{\frac{627}{49}+1} + \sqrt{\frac{627}{49}-2} = \frac{26}{7} + \frac{23}{7} = \frac{49}{7} = 7$ ✓

Same for $-\frac{\sqrt{627}}{7}$.

Two solutions $x = \pm\frac{\sqrt{627}}{7}$.

(11) Zeros: $x = 1/3$ multiplicity 2, f does not change sign

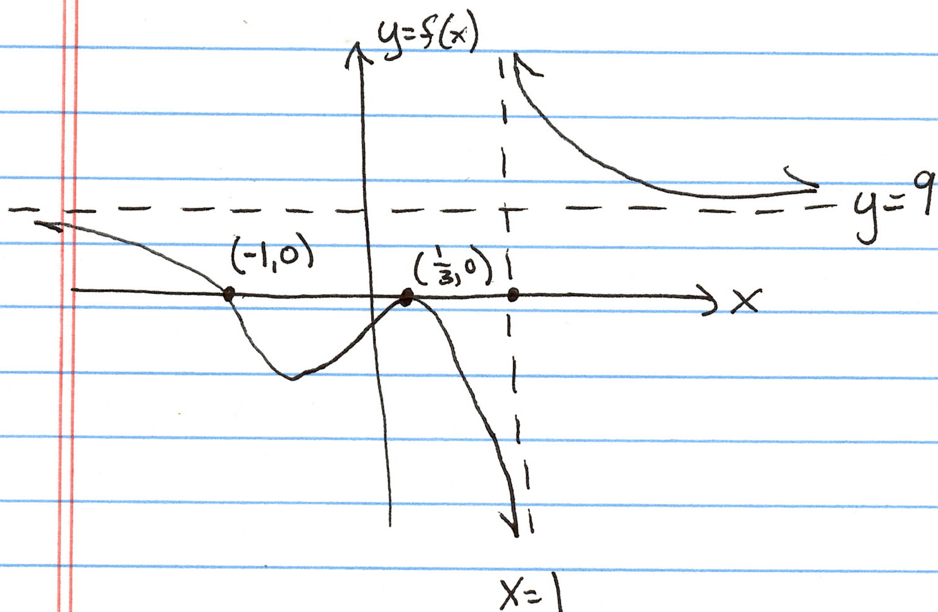
$x = -1$ multiplicity 1, f changes sign

Vertical Asymptotes: $x = 1$ multiplicity 3, f changes sign.

End Behaviour: If $|x|$ large, $f(x) \sim \dfrac{(3x)^2(x)}{(x)^3}$

$= 9$

$\rightarrow \lim_{x \to \pm\infty} [f(x)] = 9$. Horizontal Asymptote.



(12) $|2x-x^2| = 2x-4$

$2x-x^2 = 2x-4$ or $2x-x^2 = -(2x-4)$

$-x^2 = -4$ $\quad$ $-x^2+4x-4=0$

$x^2 = 4$ $\quad$ $x^2-4x+4=0$

$x = \pm 2.$ $\quad$ $(x-2)^2=0$

$x=2.$

Check $|2(2)-(2)^2| = 2(2)-4$

$0=0$ ✓

$|2(-2)-(-2)^2| = 2(-2)-4$

$|-8| = -8$

$8 = -8$ ⊗

One solution, $x=2$.

$x=-2$ is extraneous.

(13) $|2x| = 2-|x|$

$2x = 2-|x|$ or $2x = -(2-|x|)$

$|x| = 2-2x$ $\quad$ $2x = -2+|x|$

$|x| = 2x+2$

$x = 2-2x$ or $x = -(2-2x)$ or $x = 2x+2$ or $x = -(2x+2)$

$3x = 2$ $\quad$ $-x=-2$ $\quad$ $-x=-2$ $\quad$ $3x=-2$

$x = 2/3$ $\quad$ $x=2$ $\quad$ $x=2$ $\quad$ $x=-2/3$

check: $|2\cdot 2/3| = 2-|2/3|$ $\quad$ $|2\cdot 2| = 2-|2|$ $\quad$ $|2\cdot(-2/3)| = 2-|-2/3|$

$4/3 = 2-2/3$ $\quad$ $4 = 2-2$ $\quad$ $4/3 = 2-2/3$

$4/3 = 4/3$ ✓ $\quad$ $4=0$ ⊗ $\quad$ $4/3 = 4/3$ ✓

Two solutions $x = \pm 2/3$.

Note: This process works for more complicated problems, but in this case $|2x| = 2|x| \rightarrow 3|x| = 2 \rightarrow |x| = 2/3$

$x = \pm 2/3.$

(14) $f(x) = 12x^4 - 16x^3 + 36x^2 - 48x$

$= x(12x^3 - 16x^2 + 36x - 48)$

Factors -48: $\pm(1, 2, 3, 4, 6, 8, 12, 16, 24, 48)$

Factors 12: $\pm(1, 2, 3, 4, 6, 12)$

There a multitude of potential rational zeros!

But.... I could have been smarter.

~~$f(4/3)$~~

$f(x) = 4x(3x^3 - 4x^2 + 9x - 12)$

Factors of -12: $\pm(1, 2, 3, 4, 6, 12)$

Factors of 3: $\pm(1, 3)$

Possible Rational zeros: $\pm(1, 2, 3, 4, 6, 12, \frac{1}{3}, \frac{2}{3}, \frac{4}{3})$

$f(4/3) = 4(4/3)\left(3\left(\frac{4}{3}\right)^3 - 4\left(\frac{4}{3}\right)^2 + 9(4/3) - 12\right)$

$= 0.$

So $x - 4/3$ is a factor, $\longrightarrow$ $3x - 4$ is a factor.

$$
\begin{array}{r|l}
 & x^2 + 3 \\
3x-4 & 3x^3 - 4x^2 + 9x - 12 \\
 & \underline{3x^3 - 4x^2} \quad \text{subtract} \\
 & \quad 9x - 12 \\
 & \quad \underline{9x - 12} \quad \text{subtract} \\
 & \quad\quad 0
\end{array}
$$

$f(x) = 4x(3x-4)(x^2+3)$

$x = 0$

$x = 4/3$

$x^2 + 3 = 0 \longrightarrow x = \pm\sqrt{3}\,i$

(15) $x \le \dfrac{1}{(x-2)^2}$

$$x - \frac{1}{(x-2)^2} \le 0$$

$$\frac{x(x-2)^2 - 1}{(x-2)^2} \le 0$$

$$\frac{x(x^2-4x+4) - 1}{(x-2)^2} \le 0$$

$$\frac{x^3 - 4x^2 + 4x - 1}{(x-2)^2} \le 0$$

Factor numerator.

$$\begin{array}{r|l} & x^2 - 3x + 1 \\ \hline x-1 & x^3 - 4x^2 + 4x - 1 \\ & x^3 - x^2 \\ \hline & -3x^2 + 4x \\ & -3x^2 + 3x \\ \hline & x - 1 \\ & x - 1 \\ \hline & 0 \end{array}$$

$$f(x) = \frac{(x-1)(x^2-3x+1)}{(x-2)^2} \le 0$$

<u>Zeros</u> $x=1$ mult 1, f changes sign.

$$x^2 - 3x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9-4(1)(1)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$ multiplicity 1, f changes sign.

<u>Vertical Asymptotes</u> $x=2$ mult. 2, f does not change sign.

<u>End Behaviour</u> If $|x|$ large,

$$f(x) \sim \frac{(x)(x^2)}{(x)^2} = x.$$

$$\lim_{x \to \infty} [f(x)] = \infty \text{ (positive)}$$

$$\lim_{x \to -\infty} [f(x)] = -\infty \text{ (negative)}$$

x		$\frac{3-\sqrt{5}}{2}$		1		2		$\frac{3+\sqrt{5}}{2}$	
f	$-$	0	$+$	0	$-$	∞	$-$	0	$+$

$$x \in \left(-\infty, \frac{3-\sqrt{5}}{2}\right] \cup [1, 2) \cup \left(2, \frac{3+\sqrt{5}}{2}\right].$$

(16) $g(x) = \dfrac{(3x-1)^2 x}{(x-1)^2}$

<u>zeros</u> $x = 1/3$, mult 2, f does not change sign.

$x = 0$, mult 1, f changes sign.

<u>Vertical Asymptotes</u> $x = 1$, mult 2, f does not change sign.

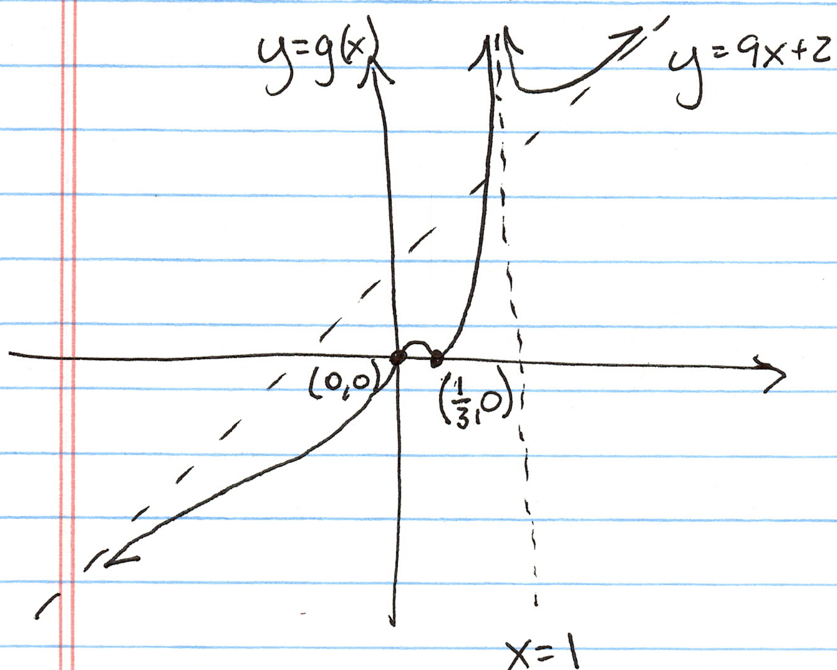
<u>End Behaviour</u> If $|x|$ is large, $g(x) \sim \dfrac{(3x)^2 x}{(x)^2} = 9x$ slant asymptote!

Get equation of slant asymptote: $g(x) = \dfrac{(9x^2-6x+1)x}{x^2-2x+1}$

$= \dfrac{9x^3-6x^2+x}{x^2-2x+1}$

$$
\begin{array}{r|l}
 & 9x + 12 \\
x^2-2x+1 & 9x^3-6x^2+x+0 \\
 & \underline{9x^3-18x^2+9x} \\
 & \quad 12x^2-8x
\end{array}
$$

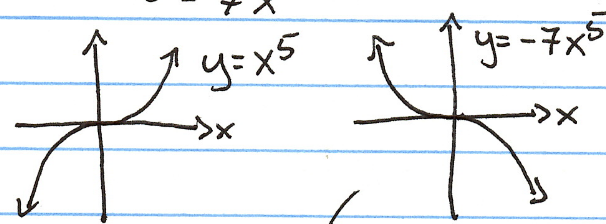
we can stop long divison once we have $y=mx+b$!



17. $f(x) = 7(2-x)^3(x^2-2x+1)$
$= 7(2-x)^3(x-1)^2$

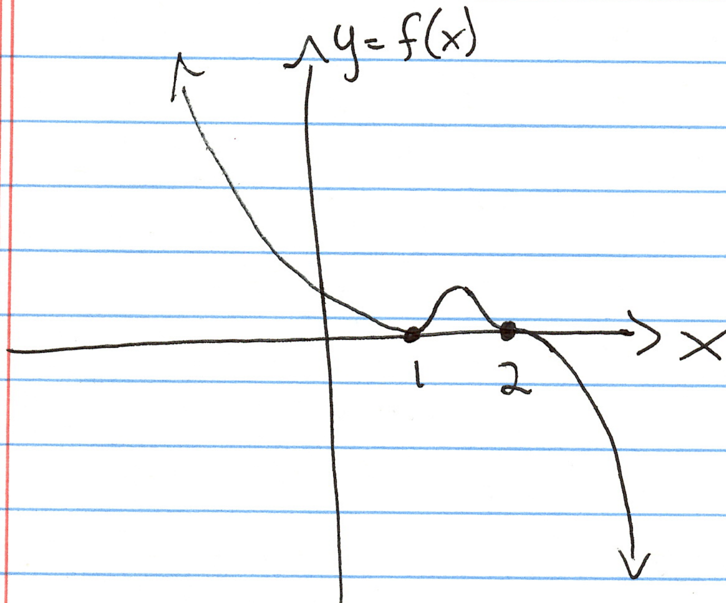
Zeros: $x=2$, mult 3, f changes sign, mult >1, so f is horizontal at $x=2$
$x=1$, mult 2, f ~~chan~~ does not change sign.

End Behaviour: If $|x|$ is large, $f(x) \sim 7(-x)^3(x^2)$
$\sim -7x^5$



$\lim_{x\to\infty} [f(x)] = -\infty$

$\lim_{x\to-\infty} [f(x)] = \infty$



(18) Let $u = x^2 + 2x$

$u^2 - u - 2 = 0$

$(u+1)(u-2) = 0$

$u = -1$ or $u = 2$

$x^2 + 2x = -1$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

$x^2 + 2x = 2$

$x^2 + 2x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}}{2}$$

$$= -1 \pm \sqrt{3}.$$

All $x = -1, -1 \pm \sqrt{3}$ solve original equation.