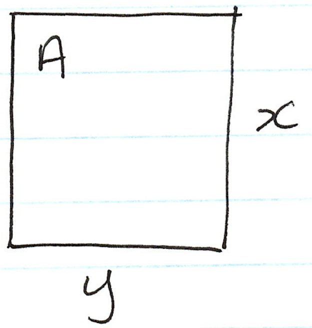
1



Perimeter $= 2x + 2y = P = 2400$ (1)

Area $= A = xy$ (2)

We want to maximize A.

Solve (1) for $y = 1200 - x$.

Therefore, $A(x) = x(1200 - x)$

$= 1200x - x^2$ domain $0 \le x \le 1200$

To maximize Area, solve $A'(x) = 0$.

$$1200 - 2x = 0$$

$$x = 600 \text{ ft.}$$

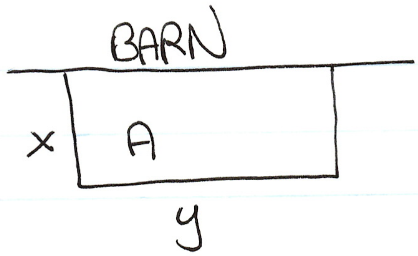
This produces a maximum, since $A''(x) = -2 < 0$ so $A(x)$ is concave down, therefore $x = 600$ gives a max by 2ND derivative test.

Dimensions $x = 600$ ft

$y = 1200 - 600$ ft $= 600$ft , a square.

Note $A(0) = A(1200) = 0$, so we have found the absolute max.

(2)



Perimeter $= P = 2x + y$ (1)

Area $= A = xy = 600\ ft^2$ (2)

We want to minimize P.

Solve (2) for $x = \frac{600}{y}$.

Therefore, $P(y) = 2\left(\frac{600}{y}\right) + y$

$$= \frac{1200}{y} + y \qquad \text{domain } y > 0$$

To minimize perimeter, solve $P'(y) = 0$

$$-\frac{1200}{y^2} + 1 = 0$$

$$y = \pm\sqrt{1200} = 20\sqrt{3}\ ft.$$

($-\sqrt{\ }$ is unphysical)

This produces a minimum, since $P''(y) = \frac{2400}{y^3}$

$$P''(20\sqrt{3}) > 0$$

so $P''(y)$ is concave up at $y = 20\sqrt{3}$, and this gives a min by 2ND derivative test. Since $P''(y) > 0$ for all $y > 0$, we have an <u>absolute min</u>.

$y = 20\sqrt{3} \sim 34.64 ft$

$x = \frac{600}{20\sqrt{3}} = \frac{30}{\sqrt{3}} = 17.32\ ft$

Buy $2x + y = \frac{60}{\sqrt{3}} + 20\sqrt{3} \cong 69.28$ ft of fencing.

(3) No sketch needed here.

Let x be first number.
y be second number.

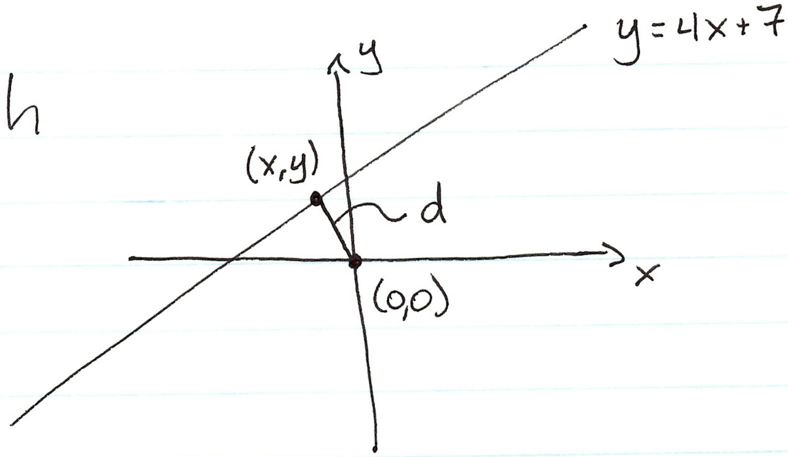
$x + y = 23$ (1) Solve (1) for $y = 23 - x$
$P = xy$ (2) Put into (2) to get
$P(x) = x(23 - x)$
$= 23x - x^2$. domain $x \in \mathbb{R}$.

Solve $P'(x) = 23 - 2x = 0$
$x = 23/2$

Since $P''(x) = -2 < 0$, $P(x)$ concave down and we have a max at $x = 23/2$ by 2^{ND} derivative test. Since $P''(x) < 0$ for all x, we have found absolute max.

$x = 23/2$
$y = 23 - 23/2 = 23/2$ are the two numbers.

(4) sketch



$d^2 = (x-0)^2 + (y-0)^2$ distance formula

$= x^2 + y^2$

$= x^2 + (4x+7)^2$ using $y = 4x+7$.

Minimize $Q(x) = x^2 + (4x+7)^2$ will minimize distance.

domain $x \in \mathbb{R}$.

$Q'(x) = 2x + 2(4x+7)(4)$

$= 56 + 34x$

Solve $Q'(x) = 0 \rightarrow x = -\frac{56}{34} = -\frac{28}{17}$.

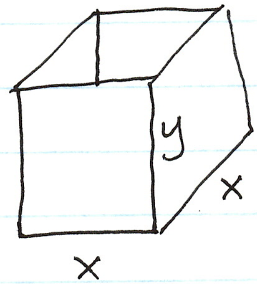
Since $Q(x)$ is a quadratic opening up, the $x = -\frac{28}{17}$ is the minimum of $Q(x)$, and will be the absolute min.

the point closest to $(0,0)$ is

$x = -28/17$

$y = 4\left(\frac{-28}{17}\right) + 7 = 7/17$

⑤ Diagram



Volume $= V = x^2 y = 32000 \text{ cm}^3$ (1)

Surface Area : $S = 4xy + x^2 \text{ cm}^2$ (2)

We want to minimize surface area.

Solve (1) for $y = \dfrac{32000}{x^2}$.

$$S(x) = 4x\left(\frac{32000}{x^2}\right) + x^2$$

$$= \frac{128000}{x} + x^2. \qquad \text{domain } x > 0$$

Solve $S'(x) = -\dfrac{128000}{x^2} + 2x = 0$

$$\frac{128000}{x^2} = 2x$$

$$64000 = x^3$$

$$x = (6400)^{1/3} = 40.$$

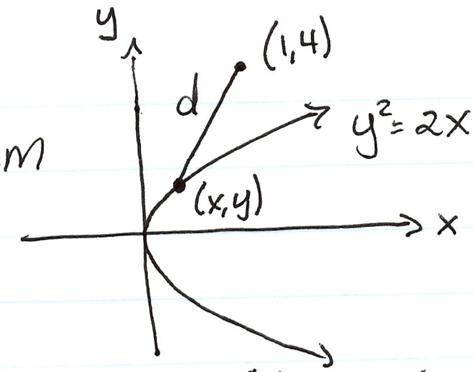
Note $S''(x) = \dfrac{2 \cdot 128000}{x^3} + 2 > 0$ for $x > 0$,

So $S(x)$ is concave up, and we have found <u>absolute min</u> on the domain by 2nd derivative test.

Dimensions $x = 40$ cm

$$y = \frac{32000}{(40)^2} = \frac{32000}{1600} = 20 \text{ cm}.$$

⑥ Diagram



$$d^2 = (1-x)^2 + (4-y)^2$$
$$= \left(1-\frac{y^2}{2}\right)^2 + (4-y)^2$$

Minimize $Q(y) = (1 - y^2/2)^2 + (4-y)^2$ Domain $y \in \mathbb{R}$.
which will minimize the distance as well.

$$Q'(y) = \cancel{2}\left(1-\frac{y^2}{2}\right)\left(-\frac{\cancel{2}y}{\cancel{2}}\right) + \cancel{2}(4-y)(-1) = 0$$

$$\cancel{-y} + \frac{y^3}{2} - 4 + \cancel{y} = 0$$

$$y^3 = 8$$
$$y = 2$$

Note: $Q'(y) = \frac{y^3}{2} - 4$

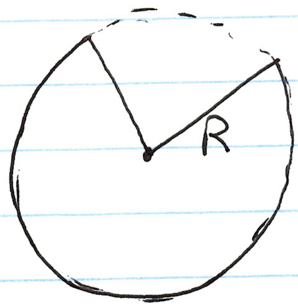
$$Q''(y) = \frac{3}{2}y^2 > 0 \text{ for all } y \in \mathbb{R}.$$

Therefore $Q(y)$ is always concave up, and we have found the absolute min of $Q(y)$ at $y=2$.

Point on $y^2 = 2x$ closest to $(1,4)$ is

$y = 2$
$x = \frac{y^2}{2} = 2.$ $\longrightarrow$ $(2,2)$.

(7)



h: height of cone

$r = \sqrt{R^2 - h^2}$

R is fixed at a constant. (Pythagorean Theorem)

We know $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi \left(\sqrt{R^2 - h^2}\right)^2 h$$

$$V(h) = \frac{1}{3}\pi \left(R^2 h - h^3\right)$$ domain $0 \le h \le R$

Maximum volume occurs when $\frac{dV}{dh} = 0$.

$$\frac{dV}{dh} = \frac{\pi}{3}\frac{d}{dh}\left[R^2 h - h^3\right]$$

$$= \frac{\pi}{3}\left(R^2 - 3h^2\right) = 0$$

$\rightarrow h = \pm \frac{R}{\sqrt{3}}$ Keep $h = \frac{R}{\sqrt{3}}$ based on domain.

$V(0) = 0$

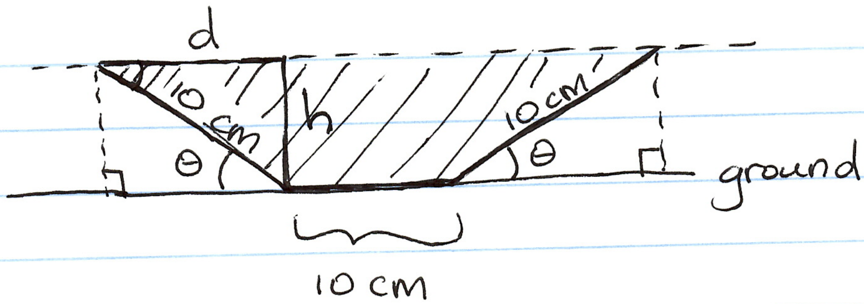
$V(R) = 0$

$$V(R/\sqrt{3}) = \frac{\pi}{3}\left(R^2 \cdot \frac{R}{\sqrt{3}} - \left(\frac{R}{\sqrt{3}}\right)^3\right) = \frac{2\pi R^3}{9\sqrt{3}}$$

So the maximum volume of cup is $\frac{2\pi R^3}{9\sqrt{3}}$

by Closed Interval Method.

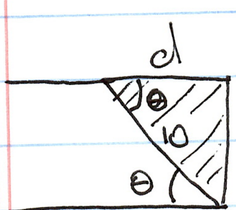
(8) Diagram



Cross section of gutter determines how much water gutter can hold.

$$\text{Area} = 10h + 2\left(\tfrac{1}{2}dh\right)$$
$$= 10h + dh$$

Note: no θ in expression!



$h = \sqrt{10^2 - d^2}$ pythagorean

not needed!

$\cos\theta = \dfrac{d}{10} \rightarrow d = 10\cos\theta$

$\sin\theta = \dfrac{h}{10} \qquad h = 10\sin\theta$

we want to maximize area $A(\theta)$.

$$A(\theta) = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta)$$
$$= 100\sin\theta\,(1 + \cos\theta)$$

domain $\theta \in [0, \pi]$ (from geometry)

$$A'(\theta) = 100\frac{d}{d\theta}[\sin\theta](1+\cos\theta) + 100\sin\theta\frac{d}{d\theta}[1+\cos\theta]$$
$$= 100(\cos\theta)(1+\cos\theta) + 100\sin\theta(-\sin\theta)$$
$$= 100\cos\theta + 100\cos^2\theta - 100\sin^2\theta.$$

~~domain $\theta \in [0, \pi]$~~

Solve $A'(\theta) = 0$

$$\cancel{100}\cos\theta + \cancel{100}\cos^2\theta - \cancel{100}\sin^2\theta = 0$$

convert everything to cosines: $\sin^2\theta = 1 - \cos^2\theta$

$$\cos\theta + \cos^2\theta - \sin^2\theta = 0$$
$$\cos\theta + \cos^2\theta - (1 - \cos^2\theta) = 0$$
$$\cos\theta + 2\cos^2\theta - 1 = 0$$

quadratic in $\cos\theta$!

8 continued

Let $u = \cos\theta$: $2u^2 + u - 1 = 0$

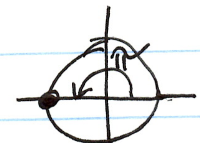
$$(2u-1)(u+1) = 0$$

$2u - 1 = 0$ or $u + 1 = 0$

$u = \frac{1}{2}$ $\qquad$ $u = -1$

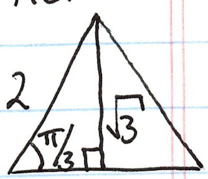
$\cos\theta = \frac{1}{2}$ $\qquad$ $\cos\theta = -1$

$\theta = \pi$



from unit circle

from



$\theta = \pi/3$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\pi/3 = \frac{\sqrt{3}}{2}$

we reference triangle to get $\sin\pi/3$

Closed Interval Method:

$$A(0) = 100\,\cancelto{0}{\sin 0}\,(1 + \cancelto{1}{\cos 0}) = 0$$

$$A(\pi) = 100\,\cancelto{0}{\sin \pi}\,(1 + \cancelto{-1}{\cos \pi}) = 0$$

$$A(\pi/3) = 100\,\cancelto{\sqrt{3}/2}{\sin(\pi/3)}\,(1 + \cancelto{1/2}{\cos \pi/3}) = 100\,\frac{\sqrt{3}}{2}\left(\frac{3}{2}\right)$$

$$= 75\sqrt{3}$$

The absolute max is $75\sqrt{3}$ at $\theta = \pi/3$ by closed Interval Method.

(9) No diagram needed.

Note $v > u$ for fish to move forward, against current.

Domain $v > u$.

$$E'(v) = \frac{d}{dv}\left[\frac{aLv^3}{v-u}\right]$$

$$= aL\frac{d}{dv}\left[\frac{v^3}{v-u}\right]$$

$$= aL\left(\frac{(v-u)\frac{d}{dv}[v^3] - v^3\frac{d}{dv}[v-u]}{(v-u)^2}\right)$$

$$= aL\left(\frac{(v-u)(3v^2) - v^3(1)}{(v-u)^2}\right)$$

$$= aL\left(\frac{2v^3 - 3v^2u}{(v-u)^2}\right)$$

$$= aLv^2\left(\frac{2v-3u}{(v-u)^2}\right)$$

$$= \frac{aLv^2(2v-3u)}{(v-u)^2}$$

Critical Numbers

Solve $E'(v) = 0$

$$\frac{aLv^2(2v-3u)}{(v-u)^2} = 0$$

$aLv^2 = 0$ or $2v - 3u = 0$

$v = 0$ $\quad$ $v = \frac{3u}{2}$.

and $E'(v)$ does not exist. which happens when $v = u$. Fish is stationary in the water.

~~Exercising~~ Exercising fish. (not in domain)

9 continued

Note: $v=0$ means fish is not swimming.
Dead Fish. (not in domain)

Left with $v = \frac{3u}{2}$. Does this maximize or minimize the energy?

Let's do a sign chart for $E'(v)$. (1st Derivative test).

Note: $v<0$ is not in domain, just a bit of fun on thinking about what these solutions mean.

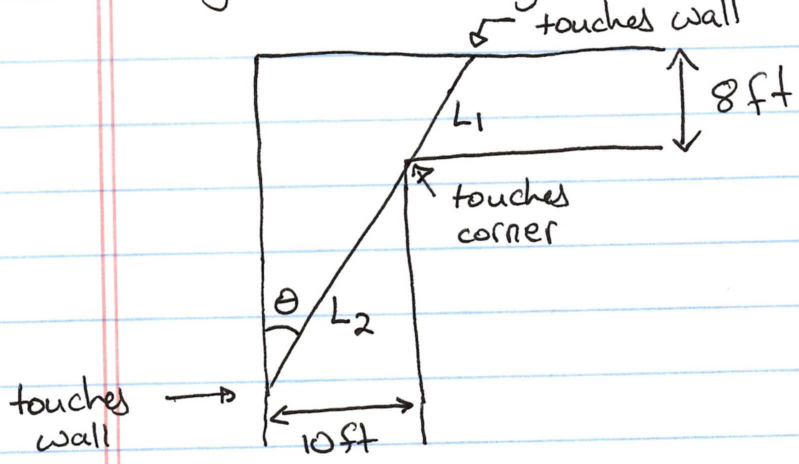
$E'(v)$: 0 at $v=0$ (dead), ∞ at $v=u$ (exercising), 0 at $v=\frac{3u}{2}$; $-$ before $\frac{3u}{2}$, $+$ after

$E(v)$: dec, inc

$E(v)$ is min at $v = 3u/2$, and it must be an absolute min based on the sign chart above (1st derivative test).

The fish should swim at a speed $\frac{3}{2}u$, where u is speed of current, to minimize energy expended.

(10) Diagram (looking from above)



Glass going around a corner.

Length of glass $= L = L_1 + L_2$.

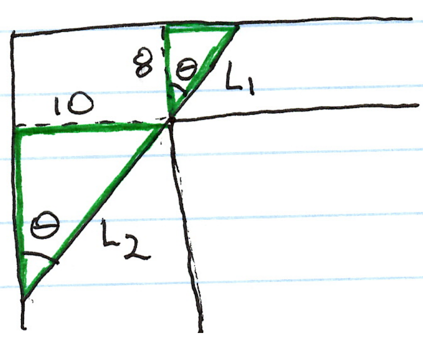
The diagram shows the situation where the glass did not make it around the corner.

If $\theta = 0$, L_2 becomes infinite.
If $\theta = \pi/2$, L_1 becomes infinite.

$L = L_1 + L_2$ must be shrinking, to the point where we have $\theta \in (0, \pi/2)$ and the glass makes the turn.

So we want to minimize L with respect to θ in the domain $\theta \in (0, \pi/2)$.

Get L_2 and L_1 in terms of θ.



$\sin\theta = \frac{opp}{hyp} = \frac{10}{L_2} \rightarrow L_2 = \frac{10}{\sin\theta}$

$\cos\theta = \frac{adj}{hyp} = \frac{8}{L_1} \qquad L_1 = \frac{8}{\cos\theta}$

10 continued

$$L(\theta) = \frac{8}{\cos\theta} + \frac{10}{\sin\theta}$$

$$= 8(\cos\theta)^{-1} + 10(\sin\theta)^{-1} \qquad \theta \in (0, \pi/2).$$

power rule and chain rule

$$L'(\theta) = \frac{d}{d\theta}\left[8(\cos\theta)^{-1} + 10(\sin\theta)^{-1}\right]$$

$$= -8(\cos\theta)^{-2}\frac{d}{d\theta}[\cos\theta] - 10(\sin\theta)^{-2}\frac{d}{d\theta}[\sin\theta]$$

$$= \frac{8\sin\theta}{\cos^2\theta} - \frac{10\cos\theta}{\sin^2\theta}$$

Solve $L'(\theta) = 0$

$$\frac{8\sin\theta}{\cos^2\theta} - \frac{10\cos\theta}{\sin^2\theta} = 0$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{10}{8}$$

$$\tan\theta = \left(\frac{5}{4}\right)^{1/3}$$

$$\theta = \arctan\left(\left(5/4\right)^{1/3}\right) \sim 0.822555 \text{ radians}$$

(need calculator to evaluate)

By hand, or use MMA to show

$$L''(\theta) = 10\cot^2(t)\csc(t) + 10\csc^3(t) + 8\sec^3(t) + 8\sec(t)\tan^2(t)$$

So $L''(\theta) > 0$ for $\theta \in (0, \pi/2)$, the first quandrant, and $L(\theta)$ is concave up on entire domain.

Therefore, we have found absolute min at $\theta = 0.822555$.

Length of glass that fits around corner

$$L(0.822555) = \frac{8}{\cos(0.822555)} + \frac{10}{\sin(0.822555)} \sim 25.4033 \text{ ft.}$$