**Concepts:** Basic Identities, Pythagorean Identities, Cofunction Identities, Even/Odd Identities.

**Basic Identities**

From the definition of the trig functions:

\[
\begin{align*}
\csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\
\sin \theta &= \frac{1}{\csc \theta} & \cos \theta &= \frac{1}{\sec \theta} & \tan \theta &= \frac{1}{\cot \theta} \\
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta}
\end{align*}
\]

**Pythagorean Identities**

Consider a point on the unit circle:

\[
\begin{align*}
P(x, y) &= (\cos \theta, \sin \theta) \\
\end{align*}
\]

which leads to triangle

Using the Pythagorean theorem, we see that (memorize this one): \( \cos^2 \theta + \sin^2 \theta = 1 \)

Derive two other identities from the one we have memorized:

Divide by \( \cos^2 \theta \):

\[
\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad \Rightarrow \quad 1 + \tan^2 \theta = \sec^2 \theta
\]

Divide by \( \sin^2 \theta \):

\[
\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \quad \Rightarrow \quad \cot^2 \theta + 1 = \csc^2 \theta
\]
Cofunction Identities

Consider the reference triangle:

\[ \begin{array}{c}
\theta \\
\beta \\
x \\
y \\
r
\end{array} \]

We have from the reference triangle:

\[
\begin{align*}
\sin \theta &= \frac{y}{r} = \cos \beta \\
\cos \theta &= \frac{x}{r} = \sin \beta \\
\tan \theta &= \frac{y}{x} = \cot \beta \\
\csc \theta &= \frac{r}{y} = \sec \beta \\
\sec \theta &= \frac{r}{x} = \csc \beta \\
\cot \theta &= \frac{x}{y} = \tan \beta \\
\end{align*}
\]

The angles must satisfy \( \theta + \beta = \frac{\pi}{2}, \beta = \frac{\pi}{2} - \theta \). Therefore,

\[
\begin{align*}
\sin \theta &= \cos \left( \frac{\pi}{2} - \theta \right) \\
\cos \theta &= \sin \left( \frac{\pi}{2} - \theta \right) \\
\tan \theta &= \cot \left( \frac{\pi}{2} - \theta \right) \\
\csc \theta &= \sec \left( \frac{\pi}{2} - \theta \right) \\
\sec \theta &= \csc \left( \frac{\pi}{2} - \theta \right) \\
\cot \theta &= \tan \left( \frac{\pi}{2} - \theta \right) \\
\end{align*}
\]

Even/Odd Identities (from sketches of trig functions)

\[
\begin{align*}
\sin (-\theta) &= -\sin \theta \\
\cos (-\theta) &= \cos \theta \\
\tan (-\theta) &= -\tan \theta \\
\csc (-\theta) &= -\csc \theta \\
\sec (-\theta) &= \sec \theta \\
\cot (-\theta) &= -\cot \theta \\
\end{align*}
\]

**Example** Use the cofunction and even/odd identities to prove \( \cos(\pi - x) = -\cos x \).

\[
\begin{align*}
\cos(\pi - x) &= \cos \left( \frac{\pi}{2} - (x - \frac{\pi}{2}) \right) \quad \text{want to use} \quad \cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta \\
&= \sin \left( x - \frac{\pi}{2} \right) \quad \text{want to use} \quad \sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta \\
&= \sin \left( -\left( \frac{\pi}{2} - x \right) \right) \quad \text{use} \quad \sin(-\theta) = -\sin \theta \\
&= -\sin \left( \frac{\pi}{2} - x \right) \\
&= -\cos (x)
\end{align*}
\]
Example Find \( \sin \theta \) and \( \tan \theta \) if \( \cos \theta = 0.8 \) and \( \tan \theta < 0 \).

We shall use trig identities rather than reference triangles, or coordinate system, which is how we would have solved this before.

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\sin^2 \theta &= 1 - \cos^2 \theta \\
\sin \theta &= \pm \sqrt{1 - \cos^2 \theta} \\
&= \pm \sqrt{1 - (0.8)^2} \\
&= \pm \sqrt{1 - 0.64} \\
&= \pm \sqrt{0.36} \\
&= \pm 0.6 \\
\end{align*}
\]

We need to figure out the correct sign.

\[
\begin{array}{c|c}
\text{I} & \text{II} \\
\hline
\text{S} & \text{A} \\
\text{T} & \text{C} \\
\text{III} & \text{IV}
\end{array}
\]

When \( \cos \theta > 0 \Rightarrow P \) is in either QI or QIV.

When \( \tan \theta < 0 \Rightarrow P \) is in either QII or QIV.

We are in Quadrant IV. In Quadrant IV, \( \sin \theta < 0 \).

Therefore, \( \sin \theta = -0.6 \).

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-0.6}{0.8} = -0.75
\]

Example Simply \( \frac{(\sec y - \tan y)(\sec y + \tan y)}{\sec y} \) to a basic trig function.

\[
\frac{(\sec y - \tan y)(\sec y + \tan y)}{\sec y} = \frac{\sec^2 y - \tan^2 y}{\sec y} \\
= \frac{1}{\sec y} \\
= \frac{\cos y}{\sec y}
\]

Example Simplify \( \sin x \cos x \tan x \sec x \csc x \) to a basic trig function:

\[
\sin x \cos x \tan x \sec x \csc x = \frac{1}{\csc x} \cdot \frac{1}{\sec x} \cdot \tan x = \tan x
\]

Example Write \( \frac{\tan^2 x}{\sec x + 1} \) as an algebraic expression involving a single trig function:

\[
\frac{\tan^2 x}{\sec x + 1} = \frac{\sec^2 x - 1}{\sec x + 1} \\
= \frac{(\sec x - 1)(\sec x + 1)}{\sec x + 1} \\
= \sec x - 1
\]
Example Find all the solutions to the equation $4 \cos^2 x - 4 \cos x + 1 = 0$.

Note this equation is quadratic in $\cos x$. Let $y = \cos x$.

\[
\begin{align*}
4 \cos^2 x - 4 \cos x + 1 &= 0 \\
4y^2 - 4y + 1 &= 0
\end{align*}
\]

\[
y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{4 \pm \sqrt{16 - 16}}{8}
\]

\[
= \frac{1}{2} \text{ multiplicity 2}
\]

\[
4y^2 - 4y + 1 = 4 \left( y - \frac{1}{2} \right)^2 = 0
\]

So now we must solve $y = \cos x = 1/2$. This comes from one of our special triangles.

\[
\begin{array}{c}
\text{hyp}=2 \\
\theta = \frac{\pi}{3} \\
\text{opp}=\sqrt{3} \\
\text{adj}=1
\end{array}
\]

Therefore, $x = \pi/3$. What other angles will be solutions?

We see the other solution is in Quadrant IV, and is $-\pi/3$.

We can also have solutions which are multiples of $2\pi$, so the solution to the original equation is $x = \pm \frac{\pi}{3} + 2k\pi, \ k = 0, \pm 1, \pm 2, \ldots$.  

Example Find all the solutions to the equation $\sqrt{2} \tan x \cos x - \tan x = 0$ in the interval $[0, 2\pi)$.

$$\sqrt{2} \tan x \cos x - \tan x = 0$$

So we have either $\tan x = 0$, or $\sqrt{2} \cos x - 1 = 0$.

Solve $\tan x = 0$:

$$\tan x = 0$$

$$\sin x = 0$$

$$\cos x = 0$$

$$x = 0, \pi$$

Solve $\sqrt{2} \cos x - 1 = 0$:

$$\left( \sqrt{2} \cos x - 1 \right) = 0$$

$$\cos x = \frac{1}{\sqrt{2}}$$

The angle comes from one of our special triangles:

The solutions are $\pi/4$ and $2\pi - \pi/4 = 7\pi/4$.

The solutions to $\sqrt{2} \tan x \cos x - \tan x = 0$ in $[0, 2\pi)$ are $x = 0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}$. 