Concepts: simple interest, compound interest, annual percentage yield, compounding continuously, mortgages

Note: These topics are all discussed in the text, but I am using slightly different language and spending more time setting up where the formulas come from using Examples/tables (which is why these notes are so long).

The mathematical concepts we use to describe finance are also used to describe how populations of organisms vary over time, how disease spreads through a population, how rumours spread through a population, even the motion of particles suspended in a fluid, as well as many other situations. Mathematics is so beautiful because the techniques you learn to solve one type of problem typically can be used to solve other problems!

Money deposited in a savings account in a bank will earn interest. The initial amount you deposit is called the principal, and the money which is earned is called the interest.

How does the money grow? What will your balance be after one year? There is a lot that goes into answering these questions, since interest can be paid in different ways.

Growth of Savings: Simple Interest

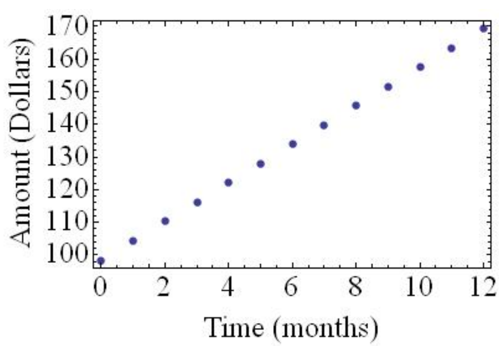
Simple interest pays interest only on the principal, not on any interest which has accumulated. Simple interest is rarely used for saving accounts, but it is used for bonds.

Example You put $98.45 in a savings account which pays simple interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution To answer this question, we can build from what we know. Simple interest means we pay interest only on the initial amount deposited (principal), which was $98.45. The interest amount will be 6%= $6/100 = 0.06$ of the principal, and added to the account balance once a month.

Interest Period	Date	Interest Added	Accumulated Amount
0	Jan 1	0	$98.45
1	Feb 1	$98.45×0.06=$5.91	$98.45 + $5.91 = $104.36
2	Mar 1	$98.45×0.06=$5.91	$104.36 + $5.91 = $110.26
3	Apr 1	$98.45×0.06=$5.91	$110.26 + $5.91 = $116.17
4	May 1	$98.45×0.06=$5.91	$116.17 + $5.91 = $122.08

This table is the form an Excel spreadsheet would take to calculate simple interest. Notice the first row is an initialization and it is the second row that contains formulas.

We see that the growth is by a constant amount ($98.45×0.06=$5.91) every time period (month in this case). This is the requirement for linear or arithmetic growth. It gets the name linear since the graph of the amount versus the time is a straight line (linear function).



Simple Interest Formula

For simple interest of r percent paid every time period with a principal P, we get

Years	Accumulated Amount
0	P
1	$(P) + Pr$
2	$(P + Pr) + Pr = P + 2Pr$
3	$(P + 2Pr) + Pr = P + 3Pr$
4	$(P + 3Pr) + Pr = P + 4Pr$
	$\vdots$
t	$P + Prt$

ie., for a principal of P with simple interest of $r\%$ paid every time period, we get an accumulated amount after t years of

$$A = P + Prt = P(1 + rt).$$

The formula gives you another way of calculating a quantity that could be done using a spreadsheet style table.

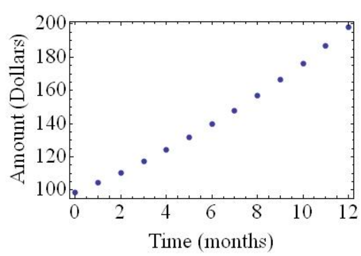
Growth of Savings: Compound Interest

Compound interest pays interest on the principal and the accumulated interest, not just the principal.

Example You put \$98.45 in a savings account which pays compound interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution To answer this question, we can build from what we know. Compound interest means we pay interest on the accumulated amount in the account. The interest amount will be 6%= $6/100 = 0.06$ of this amount, and added to the account balance once a month.

Compounding Period	Date	Interest Added	Accumulated Amount
0	Jan 1	0	\$98.45
1	Feb 1	\$98.45×0.06=\$5.91	\$98.45 + \$5.91 = \$104.36
2	Mar 1	\$104.36×0.06=\$6.26	\$104.36 + \$6.26 = \$110.62
3	Apr 1	\$110.62×0.06=\$6.64	\$110.62 + \$6.64 = \$117.26
4	May 1	\$117.26×0.06=\$7.04	\$117.26 + \$7.04 = \$124.29

We see that the amount of growth increases as time increases. The amount of growth is proportional to the amount present, which is the requirement for geometric growth.



Interest Terminology

Savings problems typically involve a bit more terminology than we've used so far.

The compounding period is the time which elapses before compound interest is paid.

The time when compounding is done effects the accumulated amount, since the current amount affects the amount of interest added, and the current amount will change if we compound more frequently.

The nominal rate is the stated rate of interest for a specified length of time. The nominal rate does not take into account how interest is compounded!

The effective rate is the actual percentage rate of increase for a length of time which takes into account compounding. It represents the amount of simple interest that would yield exactly as much interest over that length of time.

The effective annual rate (EAR) is the effective rate given over a year. For savings accounts, the EAR is also called the annual percentage yield (APY).

Compound Interest Formula

For a nominal annual rate r, compounded m times per year, we have $i = r/m$ as the interest rate per compounding period. Now let's try to derive a formula for compound interest.

Compounding Period	Amount
0	P
1	$P + Pi = P(1+i)$
2	$P(1+i) + P(1+i)i = P(1+i)^2$
3	$P(1+i)^2 + P(1+i)^2 i = P(1+i)^3$
4	$P(1+i)^3 + P(1+i)^3 i = P(1+i)^4$
$\vdots$	
n	$P(1+i)^n$

ie., for a principal of P with compound interest of $i = r/m$ paid every compounding period, we get an accumulated amount after $n = mt$ compounding periods (t is number of years, m is number of compounding periods per year) of

$$A = P(1+i)^n = P\left(1 + \frac{r}{m}\right)^{mt}.$$

Annual Percentage Yield (APY)

By definition, the APY is the simple interest rate that earns the same interest as the compound interest after one year ($t = 1$).

$$\begin{aligned} \text{Compound Interest: } A &= P\left(1+\frac{r}{m}\right)^{mt} = P\left(1+\frac{r}{m}\right)^{m} \\ \text{Simple Interest: } A &= P\left(1+rt\right) = P\left(1+\text{APY}\right) \end{aligned}$$

Set these quantities equal, and solve for APY:

$$P\left(1+\text{APY}\right) = P\left(1+\frac{r}{m}\right)^{m} \Rightarrow (1+\text{APY}) = \left(1+\frac{r}{m}\right)^{m} \Rightarrow \text{APY} = \left(1+\frac{r}{m}\right)^{m} - 1$$

Example \$1000 is deposited at 7.5% per year. Find the balance at the end of one year, and two years, if the interest paid is compounded daily. What is the APY?
Solution
The nominal annual rate is r =7.5% = 0.075, when compounded daily, means we have $m = 365$, so $i = r/m = 0.075/365 = 0.000205479$.
One year corresponds to $n = mt = 365 \times 1 = 365$, so after one year we have

$$A = P(1+i)^n = \$1000.00(1+0.000205479)^{365} = \$1077.88.$$

Two years corresponds to $n = mt = 365 \times 2 = 730$, so after two years we have

$$A = P(1+i)^n = \$1000.00(1+0.000205479)^{730} = \$1161.82.$$

$$\text{APY} = \left(1+\frac{r}{m}\right)^{m} - 1 = \left(1+\frac{0.075}{365}\right)^{365} - 1 = 7.79\%.$$

A Limit to Compounding

Sketch the graph of the accumulated amount for 10 years if the principal is $P =$\$1000 and the annual interest rate is $r =$ 10% for simple interest, compound interest compounded yearly, compound interest compounded quarterly, and compound interest compounded daily (assume 365 days in a year).

To get the values, we can use the formulas we derived. Here is the process for getting the accumulated amount after 1 year (so $t = 1$ in all formulas); the rest are calculated in a similar fashion using $t = 2, 3, 4, \ldots$.

Simple interest after 1 year:

$$A = P(1 + rt) = \$1000(1 + 0.10 \times 1) = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded yearly ($m = 1$, $i = r/m = 0.10/1 = 0.10$, and $n = mt = 1$):

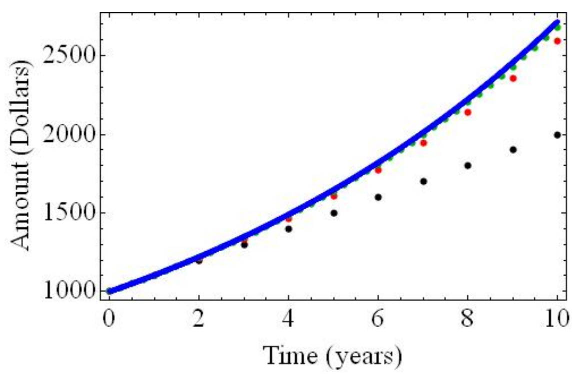
$$A = P(1 + i)^n = \$1000(1 + 0.10)^1 = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded quarterly ($m = 4$, $i = r/m = 0.10/4 = 0.025$, and $n = mt = 4$):

$$A = P(1 + i)^n = \$1000(1 + 0.025)^4 = \$1103.81 \text{ after 1 year.}$$

Compound interest compounded daily ($m = 365$, $i = r/m = 0.10/365 = 0.000273973$, and $n = mt = 365$):

$$A = P(1 + i)^n = \$1000(1 + 0.000273973)^{365} = \$1105.16 \text{ after 1 year.}$$



black: simple interest.
red: compound interest, compounded yearly.
green: compound interest, compounded quarterly.
blue: compound interest, compounded daily.

- The curves are all essentially the same for short times.
- There are more points for compounding quarterly than yearly since interest is paid more often during the year.
- There is not much difference over 10 years to compounding quarterly and compounding daily.
- Compounding more frequently leads to a larger accumulated balance, but there is a limit to this process. The limit would be if we compounded continuously.

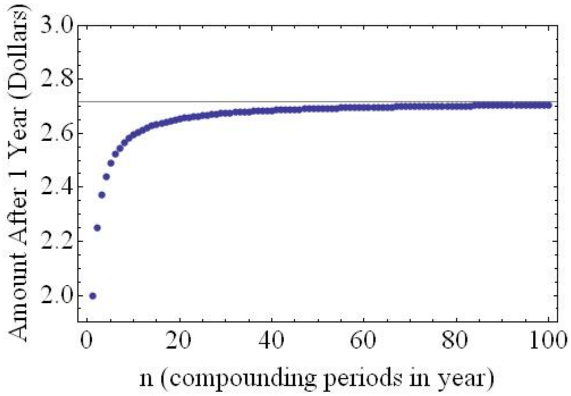
Compounding Continuously

Consider a principal $P = \$1$ and a rate of r=100% which is compounded over shorter and shorter time periods. We are interested in how much the accumulated amount will be after one year.

Compound interest compounded n times a year ($i = 1/m$, and $n = mt = m$ (to get one year, $t = 1$)):

$$A = P(1 + i)^n = \left(1 + \frac{1}{m}\right)^m \text{ after 1 year.}$$

Here is a sketch



We see that the accumulated amount is approaching a number:

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m \sim 2.71828\ldots$$

This number is similar to $\pi = 3.14...$ in that it is mathematically significant, appears in many situations, and is a nonrepeating nonterminating decimal and so we give it a special designation (not surprisingly since this was the definition given in Section 3.1!):

$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m \sim 2.71828\ldots$$

This leads the the continuous interest formula, which is

$A = Pe^{rt}$ after t years if interest is compounded continuously at annual rate r.

The continuous interest formula is the upper limit on the accumulated amount that can accrue due to compounding interest.

Review of interest formulas (principal P and annual rate r)

- Simple interest: $A = P(1 + rt)$ is the amount after t years.
- Compound interest, compounded m times over 1 year for t years: $A = P(1 + i)^n$ is the amount where $i = r/m$.
- Continuously compounded interest: $A = Pe^{rt}$ is the amount after t years.
- $\text{APY} = \left(1 + \frac{r}{m}\right)^m - 1$ is the annual percentage yield.

The formulas allow us to answer questions which would be difficult to answer using a table, and also to answer questions quickly without a lot of calculation. However, the tables allow us to answer questions that do not match the conditions under which the formulas were derived. Therefore, both formulas and spreadsheet tables are useful in understanding how personal finance works.

Accumulation: Future Value of Annuities (The Savings Formula)

An important aspect of saving is the idea of accumulation, which answers the question: *What size deposit do I have to make at regular time interval d to save a certain amount of money in a certain amount of time?* This would be important for saving for retirement, or a down payment on a house, or a car, or a child's education.

Obviously, if there was no interest, you would just break the amount you need to save into d even pieces and deposit that amount regularly. Interest makes the problem more interesting!

Example You begin saving for retirement at age 35 by paying \$100 a month into an account paying 6% annual interest compounded monthly. How much will you have in savings by the time you are 65?

Solution: The easiest way to think of this is backwards, starting by what happens at age 65. For interest compounded monthly at an annual rate of 6%, we have $i = r/m = 0.06/12 = 0.005$.
The last deposit you make will be \$100, and earn no interest (or interest for 0 months). \$100
The penultimate deposit will be \$100, and will earn interest for 1 month: $\$100(1+i)^1$.
The second last deposit will be \$100, and will earn interest for 2 month: $\$100(1+i)^2$.
This process continues, right up until the first deposit is made. In $65 - 35 = 30$ years, you will make $30 \times 12 = 360$ monthly deposits.
The amount you save is

$$A = \$100 + \$100(1+i)^1 + \$100(1+i)^2 + \cdots + \$100(1+i)^{359} = \$100\left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{359}\right].$$

We stop at 359 since we started at 0, not 1.
This is a geometric series, with $x = (1+i)$ and $n = 360$.
Therefore, we can write

$$A = \$100\left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{359}\right] = \$100\left[\frac{(1+i)^{360} - 1}{(1+i) - 1}\right] = \$100\left[\frac{(1+i)^{360} - 1}{i}\right].$$

The amount we will save by the age of 65 is

$$A = \$100\left[\frac{(1+i)^{360} - 1}{i}\right] = \$100\left[\frac{(1+0.005)^{360} - 1}{0.005}\right] = \$100\,451.50.$$

Only \$36, 000 of this is due to the deposits. The rest is interest. That should blow your mind.

The Savings Formula (Future Value of Annuity): Based on the above example, we see that for a uniform deposit d per compounding period and an interest rate of i per period, the amount A accumulated after n periods is given by:

$$A = d\left[\frac{(1+i)^n - 1}{i}\right] = d\left[\frac{(1+\frac{r}{m})^{mt} - 1}{r/m}\right]$$

Mortgages: Amortization formula (Present Value of an Annuity)

Conventional Loans

In a conventional loan, each payment pays towards the current interest that would be due over the life of the loan and also repays part of the principal. The payments are expressed in terms of an amortization table, which shows how much of each payment is going towards interest and how much towards paying off the principal.

Example You borrow \$100,000 at 8% per year for a 30 year loan for a house which will be paid off in (equal) monthly instalments. How much is your monthly payment?

Let's check this out initially using one of the many online resources: http://www.bretwhissel.net/amortization/amortize.html

We should find that the monthly payment is $d = \$733.764$.

To figure out how this monthly payment is calculated, we use both the *compound interest formula* and the *savings formula*.

Thought One: Compound Interest This situation can be thought of as borrowing the entire \$100,000 immediately, and then putting it in a savings account where it will earn interest for 30 years until it (the interest and principal) has to be repaid.

$$\begin{aligned} A &= P(1+i)^n \\ &= \$100,000(1+0.08/12)^{360} \\ &= \$1,093,572.97 \end{aligned}$$

The amount you will have saved (which is the amount that you will have to repay) in 30 years will be $1,093,572.97.

Thought Two: Savings Saving $\$d$ each month (that's what we want to find!) for 30 years means that you will save:

$$\begin{aligned} A &= d\left[\frac{(1+i)^n - 1}{i}\right] \\ &= d\left[\frac{(1+0.08/12)^{360} - 1}{0.08/12}\right] \\ &= d\left[\frac{9.93573}{0.00666667}\right] \\ &= d(1490.36) \end{aligned}$$

We want these two amounts to be exactly equal.

$$\$1,093,572.97 = 1490.36d \longrightarrow d = \$733.764$$

The monthly payments should be $733.76.

What we have done is called amortize the loan. Part of each monthly payment goes towards reducing the principal, and part goes toward reducing the interest that the loan would accumulate over the life of the loan.

Note that the million dollars itself is not what is being paid for the home, since the home is being paid off over the course of time and the principal is being reduced as time goes on.

The cost of the house is $\$733.764 \times 360 = \$264,153.60$, where $164,153.60 is due to interest. There is a small correction made at the end due to the rounding that has been done.

The Amortization Formula

Leaving things in general in the example above, we see that:

$$P(1+i)^n = d\left[\frac{(1+i)^n - 1}{i}\right]$$

where P is the principal, i is the interest rate per compounding period, and n is the number of compounding periods for which you are taking out the loan.

A little algebra can be used to rewrite this as the Amortization Formula:

$$d = P\left[\frac{i}{1-(1+i)^{-n}}\right] \qquad \text{or} \qquad d = P\left[\frac{\frac{r}{m}}{1-(1+\frac{r}{m})^{-mt}}\right]$$

The amortization formula is used to determine the monthly payments d on a conventional loan. A bit of algebra rearranges this formula into the Present Value of an Annuity formula.

Constructing The Amortization Table

First payment: $733.76:

Interest for first month on the principal is $P \times i = P \times r/m = \$100,000 \times 0.08/12 = \$666.67$.

What is left goes towards reducing the principal: $\$733.76 - \$666.67 = \$67.09$.

At the end of the first month, the principal is $\$100,000 - \$67.09 = \$99932.91$.

Second Payment: $733.76:

Interest for second month on the principal is $P \times i = P \times r/m = \$99932.91 \times 0.08/12 = \$666.22$.

What is left goes towards reducing the principal: $\$733.76 - \$666.22 = \$67.54$.

At the end of the first month, the principal is $99932.91 − $67.54 = $99865.37.

This continues until the loan is paid off. The amount you are paying per month towards principal increases, and the amount you are paying towards interest decreases.

Note: If you look at FAQ #7 on the website http://www.bretwhissel.net/blog/calculator/ you will see the need for creating your own spreadsheet for the amortization table.

Verifying the ditech.com Ad The ditech ad said the monthly payments would increase by 11% if the APR changed by 1%. Let's verify this.

Use the amortization formula: $d = P\left[\dfrac{i}{1-(1+i)^{-n}}\right]$

$P = \$200,000$
APR $= 6\% = r$
$n = 360$ (360 months = 30 years)
$i = r/12 = 0.06/12 = 0.005$
d is the monthly payment

$$
\begin{aligned}
d &= P\left[\frac{i}{1-(1+i)^{-n}}\right]\\
d &= \$200,000\left[\frac{0.005}{1-(1.005)^{-360}}\right]\\
d &= \$200,000\,[0.00599551]\\
d &= \$1199.10
\end{aligned}
$$

Redo the calculation, with APR $= r = 7\%$:
$i = r/12 = 0.07/12 = 0.00583333$

$$
\begin{aligned}
d &= P\left[\frac{i}{1-(1+i)^{-n}}\right]\\
d &= \$200,000\left[\frac{0.00583333}{1-(1.00583333)^{-360}}\right]\\
d &= \$200,000\,[0.00665302]\\
d &= \$1330.60
\end{aligned}
$$

So the percentage increase is given by $\dfrac{\$1330.60 - \$1199.10}{\$1199.10} = 0.109666 \sim 11\%$.

Interest Only Loans

The ad for Quicken Loans appears to be an interest only loan. In an interest only loan, you reduce the monthly payments by not paying anything towards principal (hence the name). Of course, this reduces your monthly payment, but at the price of building equity. Until you start paying towards the principal, you are not building any equity in your home through payments towards principal–any equity you are building is based on the fact that home prices are increasing at a rate greater than inflation. That may not always be the case, of course.

After an initial time period (could be 10 years, but it will vary) you will have substantially higher monthly payments since you need to start repaying principal, and the time for the repayment is decreased from 30 year to 20 years.

Interest only loans are not a good idea for people whose income is unlikely to change over time. They are only a good idea if you anticipate a significant jump in your income that will allow you to pay the higher payments that are required later, or if you plan on flipping the house quickly before the higher payments kick in.

Let's say we have a loan structured in the following way. P = $100,000 with an APR of r = 8% for 30 years, the first 10 of which are interest only.

For the first 10 years, you only pay interest, $P \times i = P \times r/m = \$100,000 \times 0.08/12 = \$666.67$. During this time, you have paid nothing towards the principal, so after 10 years you need an amortization loan for 20 years for the full purchase price:

$n = 12 * 20 = 240$ months
$i = r/m = 0.08/12 = 0.006666$

$$\begin{aligned} d &= P\left[\frac{i}{1-(1+i)^{-n}}\right] \\ d &= \$100,000\left[\frac{0.0066666}{1-(1.0066666)^{-240}}\right] \\ d &= \$100,000\left[0.00836435\right] \\ d &= \$836.47 \end{aligned}$$

Total Paid for Interest Only $= \$666.67 \times 120 + \$836.47 \times 240 = \$280,753$.

If we did this using a traditional 30 year fixed rate mortgage, we would find:

$n = 12 * 30 = 360$ months
$i = r/m = 0.08/12 = 0.006666$

$$\begin{aligned} d &= P\left[\frac{i}{1-(1+i)^{-n}}\right] \\ d &= \$100,000\left[\frac{0.0066666}{1-(1.0066666)^{-360}}\right] \\ d &= \$100,000\left[0.00733759\right] \\ d &= \$733.76 \end{aligned}$$

Total Paid for Traditional 30 year mortgage $= \$733.76 \times 360 = \$264,154$.

The upshot of all this is that after 30 years, you pay $\$16,600$ more interest for the interest only loan than if you had used a traditional 30 year amortization. So in the long run, an interest only loan only makes sense if

- your income is going increase dramatically in coming years, so the increased monthly payments aren't a burden,
- you can make substantial payments to the principal later on (which will reduce the amount of interest you ultimately pay), or
- you are going to sell the house before you pay it off.

These conditions are not usually satisfied by an average consumer. An average consumer should typically look for fixed rate mortgages and if they have extra money pay ahead on the principal.

The Quicken Ad If we try to verify the numbers in the Quicken Loans ad, we will find they are wrong. The monthly payment for a traditional 30 year amortization of $P = \$150,000$ at an APR of 7.5% is $d = \$1048.42$ (they got it right on their website, although they did choose to round to \$1049 instead of \$1048). The interest only payment I calculate for this loan is $P \times r/n = \$150,000 \times 0.075/12 = \$937.50$, which does not agree with the \$745 in the TV ad or \$703 on their website. We would need to look a little more closely at how this loan is structured to fully understand it.

Adjustable Rate Mortgage (ARM) ARMs have interest rates that change, and if they change in the wrong direction you might find that you cannot make the monthly payments! ARMs are generally a bad idea for an average consumer.

Check out this article from 2004, BEFORE the financial crisis:
http://www.usatoday.com/money/perfi/columnist/block/2004-06-28-mortgage_x.htm.

This is what Quicken Loans says about interest only loans today:
http://www.quickenloans.com/blog/interest-only-mortgages-the-facts.

FYI, my house **lost** \$4000 in value from 2009 to 2010.