Number Fields Ramified at One Prime

John W. Jones, Arizona State University
David P. Roberts, University of Minnesota Morris

May 18, 2008

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime,

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,
- $disc(K/\mathbf{Q}) = \pm p^a$ *for some positive integer a.*

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,
- $disc(K/\mathbf{Q}) = \pm p^a$ *for some positive integer a.*

$\mathcal{K}_{G,p}$ *is the finite (and often empty) set of G-p fields.*

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,
- $disc(K/\mathbf{Q}) = \pm p^a$ *for some positive integer a.*

$\mathcal{K}_{G,p}$ *is the finite (and often empty) set of G-p fields.*
$\mathcal{P}_G$ *is the list of primes, with each p listed $|\mathcal{K}_{G,p}|$ times.*

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,
- $disc(K/\mathbf{Q}) = \pm p^a$ *for some positive integer a.*

$\mathcal{K}_{G,p}$ *is the finite (and often empty) set of G-p fields.*
$\mathcal{P}_G$ *is the list of primes, with each p listed $|\mathcal{K}_{G,p}|$ times.*

Problems

For given G, determine

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,
- $disc(K/\mathbf{Q}) = \pm p^a$ *for some positive integer a.*

$\mathcal{K}_{G,p}$ *is the finite (and often empty) set of G-p fields.*
$\mathcal{P}_G$ *is the list of primes, with each p listed $|\mathcal{K}_{G,p}|$ times.*

Problems

For given G, determine

- *a non-empty initial segment of $\mathcal{P}_G$ and its associated fields,*

Definitions and Problems

Definitions

Given G a finite group with $|G| > 1$ and p a prime, a G-p field is a Galois number field $K \subset \mathbf{C}$ satisfying

- $Gal(K/\mathbf{Q}) \cong G$,
- $disc(K/\mathbf{Q}) = \pm p^a$ *for some positive integer* a.

$\mathcal{K}_{G,p}$ *is the finite (and often empty) set of* G-p *fields.*
$\mathcal{P}_G$ *is the list of primes, with each* p *listed* $|\mathcal{K}_{G,p}|$ *times.*

Problems

For given G*, determine*

- *a non-empty initial segment of* $\mathcal{P}_G$ *and its associated fields,*
- *the density (if it exists!) of* $\mathcal{P}_G$.

1. Abelian Groups
2. Length Two Solvable Groups
3. General Solvable Groups: The Case $G = S_4$
4. Non-Solvable Groups: A_5, S_5, A_6, S_6, G_{168}, A_7, S_7
5. $PGL_2(7)$
6. Groups of the Forms $2^r.G$ and $3.G$ for Non-Solvable G
7. Groups of the Form $2.G$
8. A Density Conjecture

1. Abelian Groups

1. Abelian Groups

The classical theory of cyclotomic fields gives complete results here.

1. Abelian Groups

The classical theory of cyclotomic fields gives complete results here.

Classical fact

$$|\mathcal{K}_{G,2}| = \begin{cases} 3 & \text{if } G \cong C_2, \\ 2 & \text{if } G \cong C_{2^a} \text{ with } a \geq 2, \\ 1 & \text{if } G \cong C_2 \times C_{2^a} \text{ with } a \geq 1, \\ 0 & \text{else.} \end{cases}$$

1. Abelian Groups

The classical theory of cyclotomic fields gives complete results here.

Classical fact

$$|\mathcal{K}_{G,2}| = \begin{cases} 3 & \textit{if } G \cong C_2, \\ 2 & \textit{if } G \cong C_{2^a} \textit{ with } a \geq 2, \\ 1 & \textit{if } G \cong C_2 \times C_{2^a} \textit{ with } a \geq 1, \\ 0 & \textit{else.} \end{cases}$$

If $p > 2$,

1. Abelian Groups

The classical theory of cyclotomic fields gives complete results here.

Classical fact

$$|\mathcal{K}_{G,2}| = \begin{cases} 3 & \textit{if } G \cong C_2, \\ 2 & \textit{if } G \cong C_{2^a} \textit{ with } a \geq 2, \\ 1 & \textit{if } G \cong C_2 \times C_{2^a} \textit{ with } a \geq 1, \\ 0 & \textit{else.} \end{cases}$$

If $p > 2$,

$$|\mathcal{K}_{G,p}| = \begin{cases} 1 & \textit{if } G \textit{ is cyclic of order } mp^a \textit{ with } m \textit{ dividing } p-1, \\ 0 & \textit{else.} \end{cases}$$

1. Abelian Groups

The classical theory of cyclotomic fields gives complete results here.

Classical fact

$$|\mathcal{K}_{G,2}| = \begin{cases} 3 & \text{if } G \cong C_2, \\ 2 & \text{if } G \cong C_{2^a} \text{ with } a \geq 2, \\ 1 & \text{if } G \cong C_2 \times C_{2^a} \text{ with } a \geq 1, \\ 0 & \text{else.} \end{cases}$$

If $p > 2$,

$$|\mathcal{K}_{G,p}| = \begin{cases} 1 & \text{if } G \text{ is cyclic of order } mp^a \text{ with } m \text{ dividing } p-1, \\ 0 & \text{else.} \end{cases}$$

Thus e.g. $\mathcal{P}_{C_{10}} = 5; 11, 31, 41, 61, 71, \ldots.$

2. Length Two Solvable Groups

A length two solvable group has the form $G = G'.G^{\mathrm{ab}}$ with G' abelian,

2. Length Two Solvable Groups

A length two solvable group has the form $G = G'.G^{\mathrm{ab}}$ with G' abelian, e.g. the quaternions $Q = C_2.C_2^2$.

If G^{ab} is cyclic, then one has a semi-direct product $G = G' : G^{\mathrm{ab}}$,

2. Length Two Solvable Groups

A length two solvable group has the form $G = G'.G^{\mathrm{ab}}$ with G' abelian, e.g. the quaternions $Q = C_2.C_2^2$.

If G^{ab} is cyclic, then one has a semi-direct product $G = G' : G^{\mathrm{ab}}$, e.g. the alternating group $A_4 = C_2^2 : C_3$.

Proposition

Let K be a G-p field with G solvable of length two. If $p \neq 2$ and $p \nmid |G'|$, then K is unramified over its maximal abelian subfield $K^{G'}$.

2. Length Two Solvable Groups

A length two solvable group has the form $G = G'.G^{\mathrm{ab}}$ with G' abelian, e.g. the quaternions $Q = C_2.C_2^2$.

If G^{ab} is cyclic, then one has a semi-direct product $G = G' : G^{\mathrm{ab}}$, e.g. the alternating group $A_4 = C_2^2 : C_3$.

Proposition

Let K be a G-p field with G solvable of length two. If $p \neq 2$ and $p \not| |G'|$, then K is unramified over its maximal abelian subfield $K^{G'}$.

Moral: For G solvable of length two, the study of G-p fields mostly reduces to the classical theory of class numbers of abelian fields.

Initial segments of some $\mathcal{P}_G$

Initial segments of some $\mathcal{P}_G$

G	G^{ab}	p_w	**Tame Primes**
S_3	2	3	23, 31, 59, 83, 107, 139, 199, 211, 229, 239, 257, 283, 307, 331
D_5	2		47, 79, 103, 127, 131, 179, 227, 239, 347, 401, 439, 443, 479, 523
D_7	2	7	71, 151, 223, 251, 431, 463, 467, 487, 503, 577, 587, 743, 811, 827
D_{11}	2	11	167, 271, 659, 839, 967, 1283, 1297, 1303, 1307, 1459, 1531, 1583
D_{13}	2		191, 263, 607, 631, 727, 1019, 1439, 1451, 1499, 1667, 1907, 2131
A_4	3		163, 277, 349, 397, 547, 607, 709, 853, 937, 1009, 1399, 1699
7:3	3		313, 877, 1129, 1567, 1831, 1987, 2437, 2557, 3217, 3571, 4219
F_5	4	$5^{(2)}$	101, 157, 173, 181, 197, 349, 373, 421, 457, 461, 613, 641, $653^{(2)}$
3^2:4	4		149, 293, 661, 733, 1373, 1381, 1613, 1621, 1733, 1973, 2861
F_7	6	7	211, 463, 487, 619, 877, 907, 991, 1069, 1171, 1231, 1303, 1381

3. General Solvable Groups: The Case $G = S_4$

3. General Solvable Groups: The Case $G = S_4$

For general solvable groups, calculations involve iterative use of ray class groups.

3. General Solvable Groups: The Case $G = S_4$

For general solvable groups, calculations involve iterative use of ray class groups.

In general, it is best to divide up the fields considered according to behavior at ∞ and behavior at p.

3. General Solvable Groups: The Case $G = S_4$

For general solvable groups, calculations involve iterative use of ray class groups.

In general, it is best to divide up the fields considered according to behavior at ∞ and behavior at p.

For $G = S_4 = C_2^2 : C_3 : C_2$, the six parts of $\mathcal{P}_G$ start like this:

3. General Solvable Groups: The Case $G = S_4$

For general solvable groups, calculations involve iterative use of ray class groups.

In general, it is best to divide up the fields considered according to behavior at ∞ and behavior at p.

For $G = S_4 = C_2^2 : C_3 : C_2$, the six parts of $\mathcal{P}_G$ start like this:

$\lambda \backslash s$	0	1	2
4	$2713, 2777^{(2)}, 2857$	$59, 107, 139, 283^{(2)}$	$229^{(2)}, 733, 1373$
211	$2777, 7537, 8069$	$283, 331, 491, 563$	$229, 257, 761$

3. General Solvable Groups: The Case $G = S_4$

For general solvable groups, calculations involve iterative use of ray class groups.

In general, it is best to divide up the fields considered according to behavior at ∞ and behavior at p.

For $G = S_4 = C_2^2 : C_3 : C_2$, the six parts of $\mathcal{P}_G$ start like this:

$\lambda \backslash s$	0	1	2
4	$2713, 2777^{(2)}, 2857$	$59, 107, 139, 283^{(2)}$	$229^{(2)}, 733, 1373$
211	$2777, 7537, 8069$	$283, 331, 491, 563$	$229, 257, 761$

Experimentally, the density depends on s in a $1 : 6 : 3$ ratio but does not depend on λ.

4. Non-Solvable G: A_5, S_5, A_6, S_6, G_{168}, A_7, S_7

4. Non-Solvable G: A_5, S_5, A_6, S_6, G_{168}, A_7, S_7

Initial segments of $\mathcal{P}_G$ established by extensive search over candidate defining polynomials.

4. Non-Solvable G: A_5, S_5, A_6, S_6, G_{168}, A_7, S_7

Initial segments of $\mathcal{P}_G$ established by extensive search over candidate defining polynomials. Results for A_6:

4. Non-Solvable G: A_5, S_5, A_6, S_6, G_{168}, A_7, S_7

Initial segments of $\mathcal{P}_G$ established by extensive search over candidate defining polynomials. Results for A_6:

Theorem

There are exactly two A_6-p fields with $p \leq 1677$. Moreover, the minimal prime for an A_6-p field with $\lambda = 2211$ is $p = 3929$.

p	λ	s	$f_{A_6,p}(x)$	$\mathrm{cl}_p(F_6)$	$\mathrm{cl}_p(F_6^t)$	cl_p
1579	42	2	$x^6 - x^5 + 41x^4 - 349x^3$ $+12x^2 + 3099x + 2851$	$2{\cdot}3{\cdot}3$	$2{\cdot}2{\cdot}3{\cdot}3$	$2{\cdot}3$
1667	42	2	$x^6 - 2x^5 - 39x^4 + 60x^3$ $+380x^2 + 1267x + 100$	$2{\cdot}3$	$2{\cdot}2{\cdot}3$	2
			$\vdots$			
3929	2211	2	$x^6 - x^5 - 3x^4 + 9x^3$ $-8x^2 + 2x - 1$	$8{\cdot}8{\cdot}3$	$8{\cdot}2{\cdot}3$	8

5. $PGL_2(7)$

The Klüners-Malle website contains the polynomial

$$f_0(x) = x^8 - x^7 + 3x^6 - 3x^5 + 2x^4 - 2x^3 + 5x^2 + 5x + 1$$

defining a $PGL_2(7)$-p field K for the remarkably small prime $p = 53$ (and ramification partition $\lambda = 611$).

5. $PGL_2(7)$

The Klüners-Malle website contains the polynomial

$$f_0(x) = x^8 - x^7 + 3x^6 - 3x^5 + 2x^4 - 2x^3 + 5x^2 + 5x + 1$$

defining a $PGL_2(7)$-p field K for the remarkably small prime $p = 53$ (and ramification partition $\lambda = 611$).

Proposition

Assuming the generalized Riemann hypothesis, K is the only $PGL_2(7)$-p field with $p \leq 53$.

5. $PGL_2(7)$

The Klüners-Malle website contains the polynomial

$$f_0(x) = x^8 - x^7 + 3x^6 - 3x^5 + 2x^4 - 2x^3 + 5x^2 + 5x + 1$$

defining a $PGL_2(7)$-p field K for the remarkably small prime $p = 53$ (and ramification partition $\lambda = 611$).

Proposition

Assuming the generalized Riemann hypothesis, K is the only $PGL_2(7)$-p field with $p \leq 53$.

Proof uses GRH bounds to eliminate all $p \leq 53$ except $p = 7$ which is eliminated unconditionally via Khare's proof of the Serre conjecture.

5. $PGL_2(7)$

The Klüners-Malle website contains the polynomial

$$f_0(x) = x^8 - x^7 + 3x^6 - 3x^5 + 2x^4 - 2x^3 + 5x^2 + 5x + 1$$

defining a $PGL_2(7)$-p field K for the remarkably small prime $p = 53$ (and ramification partition $\lambda = 611$).

Proposition

Assuming the generalized Riemann hypothesis, K is the only $PGL_2(7)$-p field with $p \leq 53$.

Proof uses GRH bounds to eliminate all $p \leq 53$ except $p = 7$ which is eliminated unconditionally via Khare's proof of the Serre conjecture.

(Could try to prove unconditionally by octic searches.)

6. Groups $2^r.G$ and $3.G$ for Non-Solvable G

Assorted results with base group $G = A_5$, S_5, A_6, and S_6.

6. Groups $2^r.G$ and $3.G$ for Non-Solvable G

Assorted results with base group $G = A_5$, S_5, A_6, and S_6. For example, consider the non-split extension $3.A_6$.

6. Groups $2^r.G$ and $3.G$ for Non-Solvable G

Assorted results with base group $G = A_5$, S_5, A_6, and S_6. For example, consider the non-split extension $3.A_6$.

Proposition

$\mathcal{P}_{3.A_6}$ begins 1579, 1579, 1579, 1667, ... The first three fields are given by

$$f_{3.A_6,1579,a}(x) = x^{18} - 6x^{17} - 23x^{16} + 211x^{15} - 283x^{14} - 115x^{13} - 2146x^{12} + 6909x^{11} - 3119x^{10} + 9687x^{9} - 35475x^{8} - 3061x^{7} + 47135x^{6} + 14267x^{5} - 13368x^{4} - 19592x^{3} - 10421x^{2} - 4728x - 297$$

and its two cubic twists $f_{3.A_6,1579,b}(x)$ and $f_{3.A_6,1579,c}(x)$.

7. Groups of the Form $2.G$

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G.

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G. Then K embeds in a $\tilde{G}$-p field $\tilde{K}$ if and only if complex conjugation $c \in G$ lifts to an order two element in $\tilde{G}$.

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G. Then K embeds in a $\tilde{G}$-p field $\tilde{K}$ if and only if complex conjugation $c \in G$ lifts to an order two element in $\tilde{G}$. Instead of working with this obstruction $\epsilon_\infty \in \{-1, 1\}$,

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G. Then K embeds in a $\tilde{G}$-p field $\tilde{K}$ if and only if complex conjugation $c \in G$ lifts to an order two element in $\tilde{G}$. Instead of working with this obstruction $\epsilon_\infty \in \{-1, 1\}$, one could work with $\epsilon_p \in \{-1, 1\}$,

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G. Then K embeds in a $\tilde{G}$-p field $\tilde{K}$ if and only if complex conjugation $c \in G$ lifts to an order two element in $\tilde{G}$. Instead of working with this obstruction $\epsilon_\infty \in \{-1, 1\}$, one could work with $\epsilon_p \in \{-1, 1\}$, as $\epsilon_\infty \epsilon_p = 1$.

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G. Then K embeds in a $\tilde{G}$-p field $\tilde{K}$ if and only if complex conjugation $c \in G$ lifts to an order two element in $\tilde{G}$. Instead of working with this obstruction $\epsilon_\infty \in \{-1, 1\}$, one could work with $\epsilon_p \in \{-1, 1\}$, as $\epsilon_\infty \epsilon_p = 1$.

For example, take $G = PGL_2(11)$. Then $\mathcal{P}_G$ begins at $p = 11$ with a field K going back to at least 1888 (Kiepert).

7. Groups of the Form $2.G$

Let K be a G-p field and let $\tilde{G} = 2.G$ be a double cover of G. Then K embeds in a $\tilde{G}$-p field $\tilde{K}$ if and only if complex conjugation $c \in G$ lifts to an order two element in $\tilde{G}$. Instead of working with this obstruction $\epsilon_\infty \in \{-1, 1\}$, one could work with $\epsilon_p \in \{-1, 1\}$, as $\epsilon_\infty \epsilon_p = 1$.

For example, take $G = PGL_2(11)$. Then $\mathcal{P}_G$ begins at $p = 11$ with a field K going back to at least 1888 (Kiepert). Likewise,

Proposition

$\mathcal{P}_{SL_{11}^{\pm}}$ *begins* 11, ... *with the first field given by*

$$f_{SL_2^{\pm}(11),11}(x) = x^{24} + 90p^2x^{12} - 640p^2x^8 + 2280p^2x^6 - 512p^2x^4 + 2432px^2 - p^3.$$

8. A Density Conjecture

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p \le x} |\mathcal{K}_{G,p}| / \sum_{p \le x} 1$ tends to a positive limit δ_G as $x \to \infty$.

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p \leq x} |\mathcal{K}_{G,p}| / \sum_{p \leq x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}_{C_m}^{\mathrm{tame}}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p \le x} |\mathcal{K}_{G,p}| / \sum_{p \le x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}_{C_m}^{\mathrm{tame}}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general.

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p\leq x} |\mathcal{K}_{G,p}| / \sum_{p\leq x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}^{\mathrm{tame}}_{C_m}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general. For $G = S_n$, it is

$$\delta_n = \frac{1}{2}\frac{1}{1+\delta_{n6}} P_n^{\mathrm{odd}} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2s)!s!2^s}$$

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p\leq x} |\mathcal{K}_{G,p}| / \sum_{p\leq x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}^{\mathrm{tame}}_{C_m}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general. For $G = S_n$, it is

$$\delta_n = \frac{1}{2}\frac{1}{1+\delta_{n6}} P_n^{\mathrm{odd}} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2s)!s!2^s}$$

with a reason for each factor:

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p\le x} |\mathcal{K}_{G,p}| / \sum_{p \le x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}_{C_m}^{\mathrm{tame}}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general. For $G = S_n$, it is

$$\delta_n = \frac{1}{2} \frac{1}{1+\delta_{n6}} P_n^{\mathrm{odd}} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2s)!s!2^s}$$

with a reason for each factor: $\epsilon_p \epsilon_\infty = 1$,

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p \le x} |\mathcal{K}_{G,p}| / \sum_{p \le x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}_{C_m}^{\mathrm{tame}}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general. For $G = S_n$, it is

$$\delta_n = \frac{1}{2} \frac{1}{1+\delta_{n6}} P_n^{\mathrm{odd}} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2s)!s!2^s}$$

with a reason for each factor: $\epsilon_p \epsilon_\infty = 1$, $|\mathrm{Out}(S_n)| = 1 + \delta_{n6}$,

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p \le x} |\mathcal{K}_{G,p}| / \sum_{p \le x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}^{\mathrm{tame}}_{C_m}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general. For $G = S_n$, it is

$$\delta_n = \frac{1}{2} \frac{1}{1 + \delta_{n6}} P_n^{\mathrm{odd}} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2s)! s! 2^s}$$

with a reason for each factor: $\epsilon_p \epsilon_\infty = 1$, $|\mathrm{Out}(S_n)| = 1 + \delta_{n6}$, the number of possibilities for λ,

8. A Density Conjecture

Conjecture

Let G be a finite group with $|G| > 1$ and G^{ab} cyclic. Then the ratio $\sum_{p \leq x} |\mathcal{K}_{G,p}| / \sum_{p \leq x} 1$ tends to a positive limit δ_G as $x \to \infty$.

Simple example: The conjecture is certainly true if G is the cyclic group C_m. In fact, $\mathcal{P}_{C_m}^{\mathrm{tame}}$ is the set of primes congruent to 1 modulo m ,and so $\delta_{C_m} = 1/\phi(m)$.

There is a natural candidate for δ_G in general. For $G = S_n$, it is

$$\delta_n = \frac{1}{2} \frac{1}{1 + \delta_{n6}} P_n^{\mathrm{odd}} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2s)! s! 2^s}$$

with a reason for each factor: $\epsilon_p \epsilon_\infty = 1$, $|\mathrm{Out}(S_n)| = 1 + \delta_{n6}$, the number of possibilities for λ, and the fraction of permutations in S_n which are involutions.

The constant δ_n

In particular, one has

n	3	4	5	6	7
δ_n	$0.\overline{3}$	$0.41\overline{6}$	0.325	$0.1319\overline{4}$	$0.16\overline{1}$

The constant δ_n

In particular, one has

n	3	4	5	6	7
δ_n	$0.\overline{3}$	$0.41\overline{6}$	0.325	$0.1319\overline{4}$	$0.16\overline{1}$

Thus e.g. $|\mathcal{K}_{S_7,p}|$ should have average size $0.16\overline{1}$.

The constant δ_n

In particular, one has

n	3	4	5	6	7
δ_n	$0.\overline{3}$	$0.41\overline{6}$	0.325	$0.1319\overline{4}$	$0.16\overline{1}$

Thus e.g. $|\mathcal{K}_{S_7,p}|$ should have average size $0.16\overline{1}$.

For general (rather than prime power) absolute discriminants, the $n = 3$ case is the Davenport-Heilbronn theorem and the $n = 4, 5$ cases are theorems of Bhargava.

The constant δ_n

In particular, one has

n	3	4	5	6	7
δ_n	$0.\overline{3}$	$0.41\overline{6}$	0.325	$0.1319\overline{4}$	$0.16\overline{1}$

Thus e.g. $|\mathcal{K}_{S_7,p}|$ should have average size $0.16\overline{1}$.

For general (rather than prime power) absolute discriminants, the $n = 3$ case is the Davenport-Heilbronn theorem and the $n = 4, 5$ cases are theorems of Bhargava. However we do not know theorems saying that the prime power case behaves similarly to the general case.

The constant δ_n

In particular, one has

n	3	4	5	6	7
δ_n	$0.\overline{3}$	$0.41\overline{6}$	0.325	$0.1319\overline{4}$	$0.16\overline{1}$

Thus e.g. $|\mathcal{K}_{S_7,p}|$ should have average size $0.16\overline{1}$.

For general (rather than prime power) absolute discriminants, the $n = 3$ case is the Davenport-Heilbronn theorem and the $n = 4, 5$ cases are theorems of Bhargava. However we do not know theorems saying that the prime power case behaves similarly to the general case.

For $n \geq 8$, the quantity δ_n decreases rapidly with n.

Evidence for the density conjecture from $n = 3, 4$

Evidence for the density conjecture from $n = 3, 4$

Computations of te Riele & Williams with S_3-p fields for the first several billion primes are supportive of the conjecture.

Evidence for the density conjecture from $n = 3, 4$

Computations of te Riele & Williams with S_3-p fields for the first several billion primes are supportive of the conjecture. For $n = 4$, we find

(s, λ)	$(0, 211)$	$(1, 211)$	$(2, 211)$	$(0, 4)$	$(1, 4)$	$(2, 4)$
10^2	.00	.03	.02	.00	.12	.02
10^3	.002	.056	.031	.013	.077	.034
10^4	.0080	.0698	.0399	.0161	.0965	.0462
10^5	.01047	.08589	.04567	.01676	.10525	.04837
10^6	.013471	.097131	.050874	.018186	.111884	.052834
∞	$.0208\overline{3}$	.125	.0625	$.0208\overline{3}$	.125	.0625

Evidence for the density conjecture from $n = 3, 4$

Computations of te Riele & Williams with S_3-p fields for the first several billion primes are supportive of the conjecture. For $n = 4$, we find

(s, λ)	$(0, 211)$	$(1, 211)$	$(2, 211)$	$(0, 4)$	$(1, 4)$	$(2, 4)$
10^2	.00	.03	.02	.00	.12	.02
10^3	.002	.056	.031	.013	.077	.034
10^4	.0080	.0698	.0399	.0161	.0965	.0462
10^5	.01047	.08589	.04567	.01676	.10525	.04837
10^6	.013471	.097131	.050874	.018186	.111884	.052834
∞	$.0208\overline{3}$	.125	.0625	$.0208\overline{3}$	.125	.0625

The experimental densities seem slowly converging to the conjectural asymptotic densities, just as in the $n = 3$ case.

Evidence for the density conjecture from $n = 3, 4$

Computations of te Riele & Williams with S_3-p fields for the first several billion primes are supportive of the conjecture. For $n = 4$, we find

(s, λ)	$(0, 211)$	$(1, 211)$	$(2, 211)$	$(0, 4)$	$(1, 4)$	$(2, 4)$
10^2	.00	.03	.02	.00	.12	.02
10^3	.002	.056	.031	.013	.077	.034
10^4	.0080	.0698	.0399	.0161	.0965	.0462
10^5	.01047	.08589	.04567	.01676	.10525	.04837
10^6	.013471	.097131	.050874	.018186	.111884	.052834
∞	$.0208\overline{3}$	.125	.0625	$.0208\overline{3}$	.125	.0625

The experimental densities seem slowly converging to the conjectural asymptotic densities, just as in the $n = 3$ case. The computation supports the principle of independence of λ.

Evidence for the density conjecture from $n = 3, 4$

Computations of te Riele & Williams with S_3-p fields for the first several billion primes are supportive of the conjecture. For $n = 4$, we find

(s, λ)	$(0, 211)$	$(1, 211)$	$(2, 211)$	$(0, 4)$	$(1, 4)$	$(2, 4)$
10^2	.00	.03	.02	.00	.12	.02
10^3	.002	.056	.031	.013	.077	.034
10^4	.0080	.0698	.0399	.0161	.0965	.0462
10^5	.01047	.08589	.04567	.01676	.10525	.04837
10^6	.013471	.097131	.050874	.018186	.111884	.052834
∞	$.0208\overline{3}$	.125	.0625	$.0208\overline{3}$	.125	.0625

The experimental densities seem slowly converging to the conjectural asymptotic densities, just as in the $n = 3$ case. The computation supports the principle of independence of λ. It supports the predicted ratio 1 : 6 : 3 for s ranging over 0, 1, 2.

Thanks for your attention!