Arboreal Dessins D'enfants

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1. Warm up activity: a tree game. Consider bipartite trees where each vertex is given a weight and the total weight of the black vertices is equal to the total weight of the white vertices.

Example.

$$
\begin{array}{ccccccccc}
 & & \underline{\mathbf{2}} & & 5 & & & & \\
 & & | & & | & & & & \\
\underline{\mathbf{1}} & — & 3 & — & \underline{\mathbf{7}} & — & 6 & — & \underline{\mathbf{8}} \\
 & & & & | & & & & \\
 & & & & 4 & & & &
\end{array}
$$

Give all edges a weight in the unique way such that the weight of each vertex is the sum of the weights of its incident edges.

Note that, trivially, *all edge-weights positive* $\Rightarrow$ *all vertex-weights positive.* The converse is far from being true, as indicated by the example.

The tree game. You are given:

A positive number n.
Positive numbers a_1, ..., a_r summing to n.
Positive numbers b_1, ..., b_s summing to n.

A classical formula says that there are $f(r, s) = r^{s-1}s^{r-1}$ bipartite trees with black vertices a_1, ..., a_r and white vertices b_1, ..., b_s.

You are asked to find those bipartite trees for which the induced weights on the edges are all positive.

Easy example. Vertices are $\underline{\mathbf{10}}$, 1, 2, 3, 4. There is only $f(1, 4) = 1^3 4^0 = 1$ tree. It has positive edge weights:

$$\begin{array}{ccccc} & & 2 & & \\ & & {}^{2}| & & \\ 3 & \overset{3}{-} & \underline{\mathbf{10}} & \overset{1}{-} & 1 \\ & & {}^{4}| & & \\ & & 4 & & \end{array}$$

In general, if r or s is 1 then there is one tree and its edge weights are positive.

Three examples with $(r,s) = (2,3)$ so that there are $f(2,3) = 2^2 3^1 = 12$ trees to consider in each case:

Example A	Example B	Example C
$\underline{\mathbf{6}},\ \underline{\mathbf{12}},\ 5,\ 10,\ 3$	$\underline{\mathbf{6}},\ \underline{\mathbf{12}},\ 3,\ 9,\ 6$	$\underline{\mathbf{6}},\ \underline{\mathbf{12}},\ 1,\ 8,\ 9$
$\begin{array}{l} \underline{\mathbf{6}} \overset{6}{-} 10 \overset{4}{-} \underline{\mathbf{12}} \overset{3}{-} 3 \\ \qquad\qquad\ {}^{5}\vert \\ \qquad\qquad\ 5 \end{array}$	$\begin{array}{l} \underline{\mathbf{6}} \overset{6}{-} 9 \overset{3}{-} \underline{\mathbf{12}} \overset{6}{-} 6 \\ \qquad\quad\ {}^{3}\vert \\ \qquad\quad\ 3 \end{array}$	$\begin{array}{l} \underline{\mathbf{6}} \overset{6}{-} 8 \overset{2}{-} \underline{\mathbf{12}} \overset{9}{-} 9 \\ \qquad\quad\ {}^{1}\vert \\ \qquad\quad\ 1 \end{array}$
$5 \overset{5}{-} \underline{\mathbf{6}} \overset{1}{-}\ 3\ \overset{2}{-} \underline{\mathbf{12}} \overset{10}{-} 10$	$3 \overset{3}{-} \underline{\mathbf{6}} \overset{3}{-} 6 \overset{9}{-} \underline{\mathbf{12}} \overset{3}{-} 9$	$1 \overset{1}{-} \underline{\mathbf{6}} \overset{5}{-} 9 \overset{4}{-} \underline{\mathbf{12}} \overset{8}{-} 8$
$5 \overset{5}{-} \underline{\mathbf{6}} \overset{1}{-} 10 \overset{9}{-} \underline{\mathbf{12}} \overset{3}{-} 3$	$3 \overset{3}{-} \underline{\mathbf{6}} \overset{3}{-} 9 \overset{6}{-} \underline{\mathbf{12}} \overset{6}{-} 6$	$1 \overset{1}{-} \underline{\mathbf{6}} \overset{5}{-} 8 \overset{3}{-} \underline{\mathbf{12}} \overset{9}{-} 9$
$3 \overset{3}{-} \underline{\mathbf{6}} \overset{3}{-}\ 5\ \overset{2}{-} \underline{\mathbf{12}} \overset{10}{-} 10$	(3 near misses	$\begin{array}{l} \underline{\mathbf{6}} \overset{6}{-} 9 \overset{3}{-} \underline{\mathbf{12}} \overset{8}{-} 8 \\ \qquad\quad\ {}^{1}\vert \\ \qquad\quad\ 1 \end{array}$
$3 \overset{3}{-} \underline{\mathbf{6}} \overset{3}{-} 10 \overset{7}{-} \underline{\mathbf{12}} \overset{5}{-} 5$	$6 \overset{6}{-} \underline{\mathbf{6}} \overset{0}{-} 9 \overset{9}{-} \underline{\mathbf{12}} \overset{3}{-} 3$	
	as $\overset{0}{-}$ can move)	
5 trees total	3 trees total	4 trees total

We've been working with abstract trees. Let's switch to planar trees, even though this does not seem natural from the point of view of the tree game. Suppose an abstract tree has vertex degrees $d_1, \ldots, d_v$. Then choosing a planar embedding involves choices at each vertex of degree ≥ 3. All together there are $\prod_i (d_i - 1)!$ embeddings.

As before there are $f(r, s) = r^{s-1} s^{r-1}$ bipartite trees with r black and s white vertices. Another classical formula says there are

$$g(r, s) \ = \ r_{s-1} s_{r-1}$$

planar such trees (Pochhammer symbol, e.g. $r_3 = r(r+1)(r+2) \geq r^3$).

The tree game theorem. *There are at most* $(v-2)!$ *planar trees which solve any given tree game with* v *vertices. Equality holds if and only if there are no planar trees with zero as an edge weight.*

2. Extreme rational functions and their moduli algebras. Consider rational functions $F(x) = f(x)/g(x)$ of degree n, thought of geometrically as maps $\mathbf{P}^1_x \to \mathbf{P}^1_y$. Counting multiplicities, $F(x)$ always has exactly $2n-2$ critical points in $\mathbf{P}^1_x$, the roots of

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Critical values are images of critical points and so there are exactly $2n-2$ of them in $\mathbf{P}^1_y$, counting multiplicities. Generically, all critical points are different and all critical values are different.

It is easy to write down rational functions such that 0 and ∞ are critical values to high multiplicity: just make $f(x)$ and $g(x)$ have multiple roots as desired. The most extreme case is when there is only one other critical value, say 1. Then a natural invariant of $F(x)$ is the partition triple $(\lambda_0, \lambda_1, \lambda_\infty)$ measuring multiplicities in the singular fibers. Riemann–Hurwitz says that λ_0, λ_1, and λ_∞ together have exactly $n+2$ parts.

In fact, given such a triple (λ_0, λ_1, λ_∞) there are only finitely many corresponding rational functions, up to pre-composition by Möbius transformations of $\mathbf{P}^1_x$.

Let $X(\lambda_0, \lambda_1, \lambda_\infty)$ be the set of such equivalence classes. To identify it, *one approach* is to solve equations to get an algebra

$$K(\lambda_0, \lambda_1, \lambda_\infty) = \mathbb{Q}[z]/m(z).$$

The algebra itself is well-defined up to unique isomorphism while the defining polynomial $m(z)$ depends on choices. The set $X(\lambda_0, \lambda_1, \lambda_\infty)$ is identified with the complex roots of m or, more canonically, with the set of embeddings $K(\lambda_0, \lambda_1, \lambda_\infty) \to \mathbb{C}$.

Random example from the ZIP code 56626: $K(56, 61^5, 261^3)$ is given by $m(z) =$

$$z^{16} - 4z^{15} + 50z^{14} - 400z^{13} + 1315z^{12} + \cdots$$

The Galois group of $m(z)$ is (unremarkably) all of S_{16}. The field discriminant of $m(z)$ is (remarkably!) $2^{29}3^{18}5^{17}7^{9}11^{3}$.

3. Dessins d'enfants. A *completely different approach* for identifying $X(\lambda_0, \lambda_1, \lambda_\infty)$ is to work diagrammatically.

Let $F(x) \in \mathbb{C}(x)$ with $[F] \in X(\lambda_0, \lambda_1, \lambda_\infty)$. Let D be the preimage in $\mathbf{P}^1_x$ of the real interval $[-\infty, 0] \subset \mathbf{P}^1_y$. Viewing $F^{-1}(-\infty)$ as black vertices and $F^{-1}(0)$ as white vertices, the set $D = F^{-1}([-\infty, 0])$ has the structure of bipartite graph.

Black valences are given by λ_∞.

White valencies are given by λ_0.

Face valencies are given by λ_1.

The set $X(\lambda_0, \lambda_1, \lambda_\infty)$ is in bijection with isotopy classes of such dessins.

ZIP code example again: One of the sixteen dessins is

Another dessin is the mirror image of this graph. There are six such chiral pairs and then four achiral dessins (corresponding to the twelve non-real and four real roots of $m(z)$).

Collapsing for efficiency. In general, the average valence of any dessin is $3n/(n+2) \approx 3$. There are various tricks for efficiently treating vertices and faces of valence ≤ 2. Today we will simply collapse faces of valence one to get weighted planar graphs. The above example becomes a weighted planar tree:

$$\begin{array}{ccccccccc} \underline{\mathbf{2}} & \overset{2}{\text{——}} & 5 & \overset{2}{\text{——}} & \underline{\mathbf{6}} & \overset{4}{\text{——}} & 6 & \overset{1}{\text{——}} & \underline{\mathbf{1}} \\ & & 1\,| & & & & 1\,| & & \\ & & \underline{\mathbf{1}} & & & & \underline{\mathbf{1}} & & \end{array}$$

All sixteen dessins can be obtained from the tree game, although now the 1's in λ_∞ are indistinguishable. (Connection with the tree game theorem is $(7-2)!/3!-4 = 20-4 = 16$.)

4. The arboreal case. It is well-known that both the algebraic approach and the geometric approach become easier when λ_1 is the trivial partition n. On the dessin side, this is exactly when the dessins D are trees.

In fact, the natural condition is that λ_1 has the shape of a hook, i.e. $\lambda_1 = e1^{n-e}$ for some e. On the dessin side, this is exactly when the weighted graphs representing D are weighted trees with e edges and $v = e + 1$ vertices.

For example, take $v = 4$ so that the partition triples considered have the form

$$(a_1 \cdots a_r, 31^{n-3}, b_1, \ldots, b_s).$$

Write $a_{r+i} = -b_i$ so that λ_0 and λ_∞ are captured by the vector (a_1, a_2, a_3, a_4) with $a_1 + a_2 + a_3 + a_4 = 0$. The rational functions to be considered are

$$F(x) = \prod_{i=1}^{4} (x - z_i)^{a_i}$$

with as yet unspecified z-values.

All the $F(x) = \prod_{i=1}^{4}(x-z_i)^{a_i}$ satisfy $F(\infty) = 1$. If say a_3 and a_4 are different from the other a_i we can normalize by setting $z_3 = 0$ and $z_4 = 1$ so that $F(0), F(1) \in \{0, \infty\}$. Differentiating, one has

$$F'(x) = \frac{u_0 + u_1 x + u_2 x^2}{x(x-1)(x-z_1)(x-z_2)}$$

with

$$\begin{aligned} u_0 &= -a_3 z_1 z_2 \\ u_1 &= (a_2+a_3)z_1 + (a_1+a_3)z_2 - (a_1+a_2)z_1 z_2 \\ u_2 &= a_1 z_1 + a_2 z_2 - (a_1+a_2+a_3). \end{aligned}$$

Already, $F^{-1}(1)$ contains ∞. To make $F^{-1}(1)$ contain ∞ with multiplicity three, we need to choose z_1 and z_2 so that $u_1 = u_2 = 0$. We eliminate z_1, set $(z, a, b, c) = (z_1, a_1, b_1, c_1)$ and find the moduli polynomial

$$\begin{aligned} m(a,b,c,z) &= a(a+b)z^2 - 2a(a+b+c)z \\ &\quad + (a+c)(a+b+c). \end{aligned}$$

The discriminant of this polynomial is

$$D(a,b,c) = -4abc(a+b+c).$$

Similarly, for $v = 5$ one gets the universal moduli polynomial $m(a, b, c, d, z) =$

$$
\begin{aligned}
& d^2(b+d)(c+d)(b+c+d)^2 z^6 \\
& +6ad^2(b+d)(c+d)(b+c+d)z^5 \\
& +3ad^2\left(b^3+ab^2+cb^2+2db^2+c^2b+d^2b\right. \\
& \quad +5acb+6adb+2cdb+c^3+ac^2 \\
& \quad \left.+5ad^2+cd^2+2c^2d+6acd\right)z^4 \\
& +2ad\left(cb^3+2c^2b^2+3acb^2+6adb^2+3cdb^2\right. \\
& \quad +c^3b+3ac^2b+6ad^2b+2cd^2b+2a^2cb \\
& \quad +6a^2db+3c^2db+6acdb+10a^2d^2 \\
& \quad \left.+6acd^2+6ac^2d+6a^2cd\right)z^3 \\
& +3a^2d\left(b^3+2ab^2+cb^2+db^2+a^2b+c^2b+\right. \\
& \quad 2acb+6adb+5cdb+c^3+2ac^2+ \\
& \quad \left.a^2c+5a^2d+c^2d+6acd\right)z^2 \\
& +6a^2(a+b)(a+c)(a+b+c)dz \\
& +a^2(a+b)(a+c)(a+b+c)^2
\end{aligned}
$$

with discriminant $D_5(a, b, c, d) =$

$$
\begin{aligned}
& -2^6 3^6 a^{10} b^4 c^4 d^{10} \\
& (a+b)(a+c)(b+c)(a+d)^3(b+d)(c+d) \\
& (a+b+c)^3(a+b+d)(a+c+d)(b+c+d)^3 \\
& (a+b+c+d)^{10}(b-c)^6.
\end{aligned}
$$

Similarly, $v = 6$ yields $m_6(a,b,c,d,e,z)$ of degree $4! = 24$ in both the parameters $\{a,b,c,d,e\}$ and the variable z. It has 78184 terms. In general $m_v(a_1,\ldots,a_e,z)$ has degree $(v-2)!$ in both $\{a_1,\ldots,a_e\}$ and z.

Discriminant theorem. *The polynomial discriminant $D_I(a_1,\ldots,a_r)$ has the form*

$$\pm \prod_{p \leq v-2} p^* \quad \cdot \quad \prod_{I \neq \emptyset}^{\{1,\ldots,e\}} (\sum_{i \in I} a_i)^* \quad \cdot \quad F(a_1,\ldots,a_e)^2.$$

The factor $F(a_1,\ldots,a_e)$ does not contribute to field discriminants of specializations.

5. The discriminant arrangement. The set of allowed indices for arboreal dessins with v vertices and $e = v - 1$ edges is

$$\begin{aligned} \mathbb{Z}_0^v &= \{(a_1, \dots, a_e, a_v) \in \mathbb{Z}^v : \sum a_i = 0\} \\ &\cong \{(a_1, \dots, a_e)\} \cong \mathbb{Z}^e. \end{aligned}$$

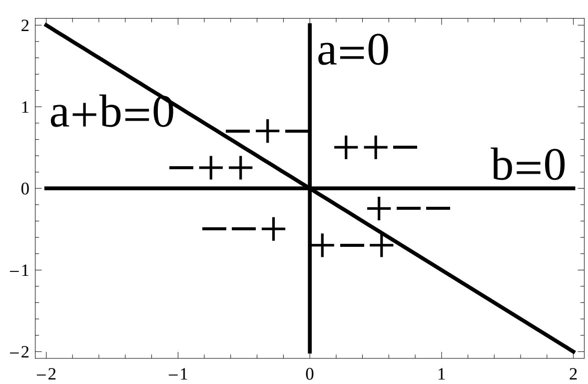
We consider this lattice inside of $\mathbb{R}_0^v \cong \mathbb{R}^e$.

For I a non-empty subset of $\{1, \dots, e\}$, let

$$H_I = \{(a_1, \dots, a_e) : \sum_{i \in I} a_i = 0\}.$$

Then the singular indices are exactly those which lie on one of these $2^e - 1$ hyperplanes.

Example of $v = 3$. Three lines in $\mathbb{R}^2$, permuted transitively by S_3:

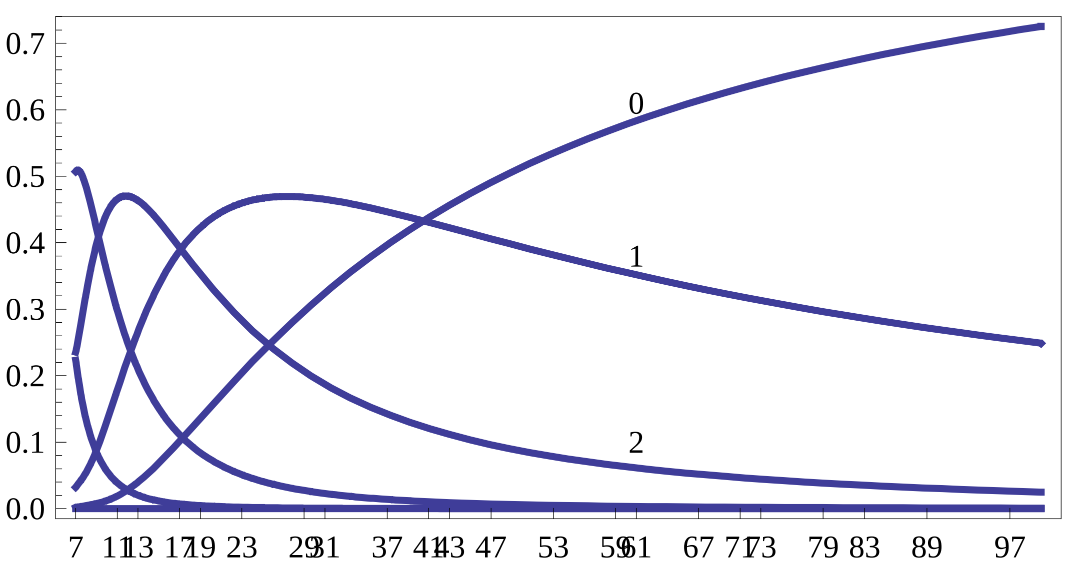


Let C_v be the set of chambers of $\mathbb{R}^v_0 - \cup_I H_I$. Then the number of chambers $|C_v|$ grows very rapidly with v. Even the number of S_v-orbits $|C_v/S_v|$ grows rapidly. For $v > 2$, it is twice the number of $\{\pm 1\} \times S_v$-orbits. These latter orbits index essentially different generic tree games with v vertices.

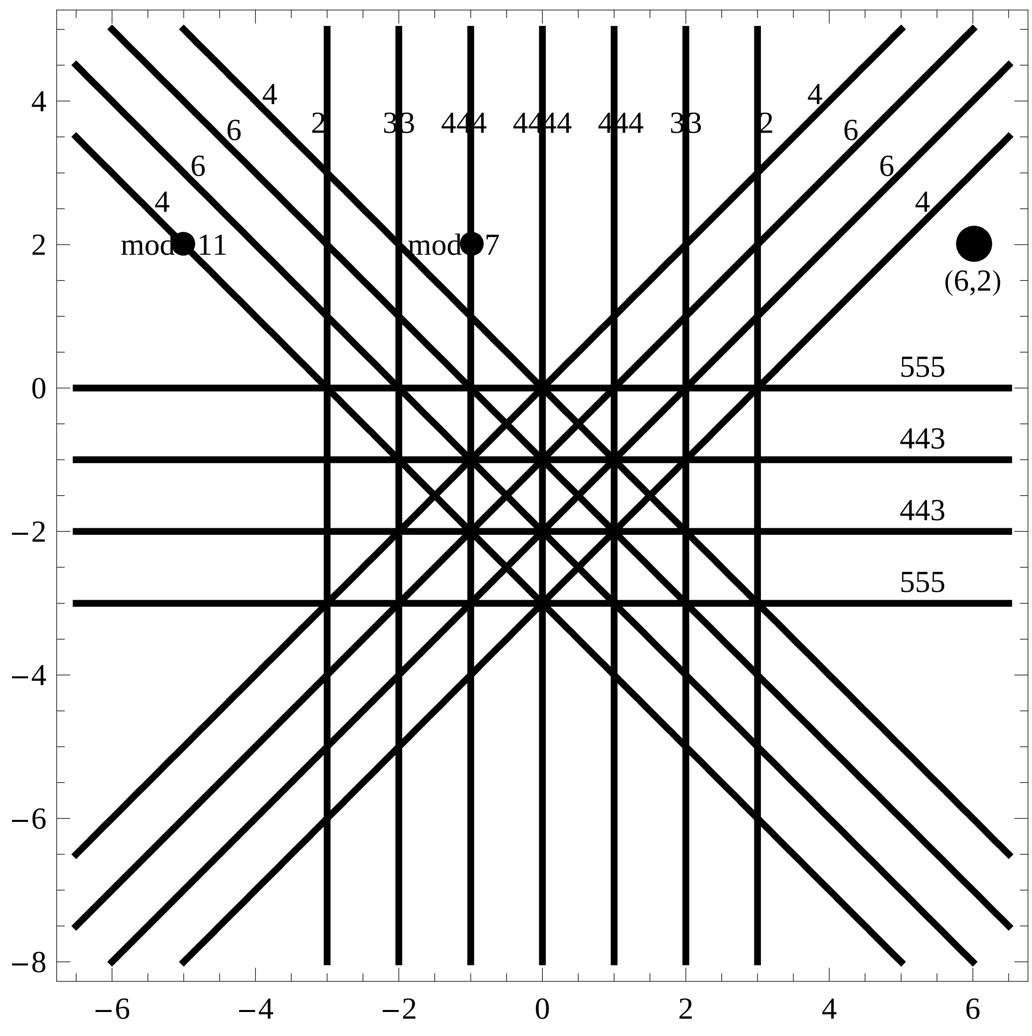
v	$\lvert C_v\rvert$	$\lvert C_v/S_v\rvert$	Distribution by Signature (r,s)
2	2	1	1
3	6	2	1 1
4	32	4	1 2 1
5	370	12	1 5 5 1
6	11292	56	1 14 26 14 1
7	1066044	576	1 62 225 225 62 1
8	347326352	8320	1 566 4059 7388 4059 566 1

Compared to many classical arrangements, our arrangement is a "thicket" of hyperplanes. For small primes p, most points in $(\mathbb{F}_p^v)_0$ are on a high-codimension stratum:

3	0						1
	1					$3p$	-3
	2				p^2	$-3p$	2
4	0						1
	1					$9p$	-9
	2				$7p^2$	$-24p$	17
	3			p^3	$-7p^2$	$15p$	-9
5	0						1
	1					$45p$	-45
	2				$55p^2$	$-240p$	185
	3			$15p^3$	$-135p^2$	$365p$	-245
	4		p^4	$-15p^3$	$80p^2$	$-170p$	104
6	0						1
	1					$625p$	-625
	2				$845p^2$	$-5340p$	4495
	3			$285p^3$	$-3705p^2$	$14055p$	-10635
	4		$31p^4$	$-660p^3$	$4990p^2$	$-14610p$	10249
	5	p^5	$-31p^4$	$375p^3$	$-2130p^2$	$5270p$	-3485



There are many natural subspaces of $\mathbb{R}_0^v$ as one can demand *symmetry relations* and/or *degeneracy relations*. For example, a natural ambient space for our ZIP code example $(56, 61^5, 261^3)$ is $(ab, 61^5, cbd^3)$. The bc-plane corresponding to the $d = 1$ slice illustrates the discriminant locus of a degree 16 polynomial:



6. Complex monodromy. Let

$$\begin{aligned} a(0) &= (a_1(0), \ldots, a_v(0)) \\ \text{and } a(1) &= (a_1(1), \ldots, a_v(1)) \end{aligned}$$

be in adjacent chambers of $\mathbb{R}^v_0 - \cup_I H_I$. Let $a(t) = (1-t)a(0) + ta(1)$ run over the segment between them. Consider trying to move solutions of the tree game at $a(0)$ to solutions of the tree game at $a(1)$.

Typically, most solutions move without any problem, meaning that edge weights stay positive for all $t \in [0,1]$. However some solutions acquire zero as an edge weight on a single edge. Nonetheless, monodromy gives two bijections $\gamma_+, \gamma_- : \mathcal{T}_0 \to \mathcal{T}_1$ corresponding to transporting in $\mathbb{C}^v_0$ either over or under the wall.

Monodromy Theorem. *On degenerating trees, γ_+ acts by moving the degenerating edge counterclockwise one spot on each component, as indicated by the following pictures. Likewise, γ_- moves the degenerating edge clockwise one spot.*

γ_+ in general:

γ_+ in Examples A,B,C:

$$t < 1/2: \quad (3{+}6t) \overset{3+6t}{\text{——}} \underline{\mathbf{6}} \overset{3-6t}{\text{——}} (5{-}4t) \overset{2+2t}{\text{——}} \underline{\underline{\mathbf{12}}} \overset{10-2t}{\text{——}} (10{-}2t) \quad \in \mathcal{T}_0$$

$$t = 1/2: \quad 6 \overset{6}{\text{——}} \underline{\mathbf{6}} \qquad 3 \overset{3}{\text{——}} \underline{\underline{\mathbf{12}}} \overset{9}{\text{——}} 9$$

$$t > 1/2: \quad (3{+}6t) \overset{6}{\text{——}} \underline{\mathbf{6}} \qquad (5{-}4t) \overset{5-4t}{\text{——}} \underline{\underline{\mathbf{12}}} \overset{10-2t}{\text{——}} (10{-}2t) \quad \in \mathcal{T}_1$$

$$t < 1/2: \quad (3{+}6t) \overset{3+6t}{\text{——}} \underline{\mathbf{6}} \overset{3-6t}{\text{——}} (10{-}2t) \overset{7+4t}{\text{——}} \underline{\underline{\mathbf{12}}} \overset{5-4t}{\text{——}} (5{-}4t) \quad \in \mathcal{T}_0$$

$$t = 1/2: \quad 6 \overset{6}{\text{——}} \underline{\mathbf{6}} \qquad 9 \overset{9}{\text{——}} \underline{\underline{\mathbf{12}}} \overset{3}{\text{——}} 3$$

$$t > 1/2: \quad (3{+}6t) \overset{6}{\text{——}} \underline{\mathbf{6}} \qquad (10{-}2t) \overset{10-2t}{\text{——}} \underline{\underline{\mathbf{12}}} \overset{5-4t}{\text{——}} (5{-}4t) \quad \in \mathcal{T}_1$$

Now suppose V is a single chamber in $\mathbb{R}_0^v$ containing a point $a(0)$ and H_I is a hyperplane bounding it. Then we can leave V via γ_+, go around H_I, and come back via γ_-^{-1}. This gives a permutation $\gamma = \gamma_-^{-1}\gamma_+$ on $\mathcal{T}_0$.

Set $i = |I|$ and $j = v - i$, with $i,\ j > 1$ to simplify. Then at the boundary H_I, there are $(i-2)!$ possibilities for one component and $(j-2)!$ possibilities for the other. They can be attached to each other in $(i-1)(j-1)$ possible ways. Let $u = \mathrm{LCM}(i-1, j-1)$. The monodromy theorem says that *the cycle structure of γ is $(i-1)!(j-1)!/u$ cycles of length u, and the rest cycles of length* 1.

One can also blow up strata of the arrangement to get a divisor with normal crossings, getting k commuting permutations on $\mathcal{T}_0$ corresponding to generating to a codimension k corner of V. This is combinatorially complicated and the explicit formulas involve Bell numbers.

7. Tame ramification. For primes $p \leq v-2$, ramification in the moduli algebras

$$K(\lambda_0, e1^{n-e}, \lambda_\infty)$$

is typically wild. For $p \geq v-1$, ramification can be described combinatorially by transporting the results from characteristic zero.

For example, the ZIP code field from the setting $v = 7$ has degree sixteen and field discriminant $2^{29}3^{18}5^{17}7^{9}11^{3}$. Thus 2, 3, and 5 are indeed wildly ramified, as is typical. The exponent 9 on 7 comes from ramification partition $4^3 1^4$. The exponent 3 on 11 comes from the ramification partition 41^{12}.

Selected References.

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