CHEBYSHEV COVERS AND EXCEPTIONAL NUMBER FIELDS

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Abstract. We define rational functions $T_{m,n}(x)$ and $U_{m,n}(x)$ in $\mathbf{Q}(x)$ by simple explicit formulas involving the classical Chebyshev polynomials $t_w(x)$ and $u_w(x)$. We show that these functions, viewed as covers of one projective line by another, are very remarkable from the point of view of Grothendieck's theory of *dessins d'enfants*. The fibers of these covering maps are likewise very remarkable from the point of view of algebraic number theory. For example, the fiber of $U_{125,128}$ above 5 is given by a degree 15875 polynomial in $\mathbf{Z}[x]$ with discriminant $-2^{130729}5^{63437}$ and Galois group the entire symmetric group S_{15875}.

1. Introduction

The Chebyshev covers of our title are the rational functions

$$T_{m,n}(x) = \frac{T_{m/2}(x)^n}{T_{n/2}(x)^m}, \qquad U_{m,n}(x) = \frac{U_{m/2}(x)^{2n}}{U_{n/2}(x)^{2m}} \tag{1.1}$$

indexed by positive integers m and n. Here $T_w(x)$, $U_w(x) \in \mathbf{Z}[x, \sqrt{x+2}, \sqrt{x-2}]$ are completions by factors of $\sqrt{x+2}$ and/or $\sqrt{x-2}$ of the classical Chebyshev polynomials, as explained in Section 2. Square roots cancel so that $T_{m,n}(x)$ and $U_{m,n}(x)$ are always in $\mathbf{Q}(x)$. The theory quickly reduces to the cases where m and n are relatively prime with $m < n$ and, in the U case, not both odd. Henceforth we restrict to these cases. We use the word "cover" because our main point of view is that the $T_{m,n}$ and $U_{m,n}$ are functions from the complex projective line with coordinate x to another complex projective line with coordinate s.

It is often convenient to clear denominators and work with polynomials. Accordingly, let

$$T_{m,n}(x) = \frac{T_{m,n}(0,x)}{T_{m,n}(\infty,x)}, \qquad U_{m,n}(x) = \frac{U_{m,n}(0,x)}{U_{m,n}(\infty,x)}, \tag{1.2}$$

in lowest terms, with monic numerator and denominator in $\mathbf{Z}[x]$. For $s \in \mathbf{C}$, let

$$T_{m,n}(s,x) = T_{m,n}(0,x) - sT_{m,n}(\infty,x), \tag{1.3}$$

$$U_{m,n}(s,x) = U_{m,n}(0,x) - sU_{m,n}(\infty,x). \tag{1.4}$$

Then the fibers above s are given as the roots of the corresponding polynomials.

A fundamental aspect of our objects is that degrees grow quadratically with the indices. If m and n have opposite parity, then $T_{m/2}(x)^n$ and $T_{n/2}(x)^{m/2}$ are already in $\mathbf{Z}[x]$. There is no cancellation between numerator and denominator and so $T_{m,n}$, considered either as a rational function or a family of polynomials, has degree $mn/2$. Otherwise there is cancellation and $T_{m,n}$ has degree $m(n-1)/2$ and $U_{m,n}$ has degree $m(n-1)$.

Sections 3-5 establish facts about Chebyshev covers. The main critical values of these covers are $s = 0$, $s = 1$, and $s = \infty$. Besides the obvious critical points in the preimages of these critical values, we show that there are $\lfloor (k-1)/2 \rfloor$ other critical points, with k always $n-m$ throughout this paper. The main result of these sections is Theorem 4.1, which gives the polynomial discriminant of $T_{m,n}(s,x)$ and $U_{m,n}(s,x)$. This theorem says in particular that most of the bad reduction of Chebyshev covers—all of it if $k = 1, 2$—is at primes dividing mn.

Sections 6-8 continue our study of Chebyshev covers. We restrict attention to the cases $k = 1, 2$, so that all ramification is above 0, 1 and ∞. For these k, we work in the greater generality of quasiChebyshev covers, defined to be three point covers having ramification partitions λ_0, λ_1, and λ_∞ agreeing with those of a Chebyshev cover. The main result of these sections is Theorem 8.1. It says that the monodromy group of a quasiChebyshev cover without non-trivial automorphisms is the full alternating or symmetric group on the degree.

Sections 9-12 concern the algebras $\mathbf{Q}[x]/T_{m,n}(s,x)$ and $\mathbf{Q}[x]/U_{m,n}(s,x)$ and closely related number fields. Section 9 concerns p-adic ramification, finding that it is highly regular and not as wild as allowed by degrees. Section 10 discusses cases with $s = \pm 1$ where Galois groups are mysteriously imprimitive. Sections 11 and 12 provide examples of number fields which are exceptional in the sense of [7]. The most extreme example is given by the polynomial $U_{125,128}(5,x)$ which has degree $125(128-1) = 15875$. Its discriminant is $-2^{130729}5^{63437}$ and the Galois group of its splitting field K is all of S_{15875}. Readers interesting in getting as quickly as possible to exceptional number fields can simply check our conventions with regard to Chebyshev polynomials in Section 2 and then skip immediately to Sections 11 and 12.

An interesting comparison with Chebyshev covers is given by covers $A_{m,n}(x) = k^{-k}x^m(n-mx)^k$ coming from trinomials [6, Section 10]. To obtain the simplest comparison, we still impose our standing conventions $(m,n) = 1$ and $k = n-m > 0$. In analogy with our Theorem 4.1, for $A_{m,n}(s,x) = x^m(n-mx)^k - sk^k$ one has the discriminant formula $D(A_{m,n}(s,x)) = \pm n^n m^{k(n-1)} k^{k(n-1)} s^{n-2}(s-1)$, so that bad reduction is at primes dividing mnk. In analogy with Theorem 8.1, the monodromy group is the full symmetric group S_n. The p-adic behavior of specialized number fields is highly regular, with $A_{m,n}(s,x)$ being studied in [4] for $s \in \mathbf{Q} - \{0,1\}$ and $A_{m,n}(1,x)/(x-1)^2$ being studied in [1].

An obvious difference between our Chebyshev covers and trinomial covers is that the degree of the latter is only n. Related to this relatively low degree is the fact that $A_{m,n}$ is rigid in the sense that it is determined by its ramification partitions $\lambda_0 = (m,k)$, $\lambda_1 = (2,1,\ldots,1)$, and $\lambda_\infty = (n)$. In other words, there are no quasitrinomial covers besides the trinomial covers. In fact, $A_{m,n}$ is linearly rigid in the strong sense of Katz [3], and this forces bad reduction at exactly primes dividing mnk. In contrast, Chebyshev covers fail badly to be linearly rigid, and there is at present no conceptual reason for their good reduction.

Let S be a finite set of primes and consider the problem raised in [5] of constructing number fields ramified within S and with Galois group A_N or S_N with N as large as possible. Specialization of covers is thus far the only systematic technique. Previous to this paper, the family of trinomial covers was the most useful infinite family of covers with respect to this constructional problem. Our $T_{m,n}$ and $U_{m,n}$

here are very much better. A natural question is whether there are other families of three point covers qualitatively similar to our Chebyshev covers.

2. Chebyshev polynomials

We work in the biquadratic extension of $\mathbf{Z}[x]$ obtained by adjoining $\sqrt{x-2}$ and $\sqrt{x+2}$. Let $w \in \{1/2, 1, 3/2, 2, \dots\}$. Our Chebyshev polynomials of the first and second kind respectively are $T_w(x)$, $U_w(x) \in \mathbf{Z}[x, \sqrt{x-2}, \sqrt{x+2}]$ where

$$T_w(z+1/z) = z^w + z^{-w}, \qquad U_w(z+1/z) = z^w - z^{-w}.$$

These Chebyshev polynomials factor canonically into their interior and boundary parts

$$T_w(x) = t_w^*(x)t_w(x), \qquad U_w(x) = u_w^*(x)u_w(x). \tag{2.1}$$

Here the boundary parts $t_w^*(x)$ and $u_w^*(x)$ depend only on their index w modulo one and are

$$t_0^*(x) = 1, \qquad u_1^*(x) = \sqrt{x^2-4}$$
$$t_{1/2}^*(x) = \sqrt{x+2}, \qquad u_{1/2}^*(x) = \sqrt{x-2}.$$

The interior parts are monic polynomials in $\mathbf{Z}[x]$ and have all roots in $(-2, 2)$. Explicitly, if w is integral,

$$t_w(x) = \sum_{j=0}^{\lfloor w/2 \rfloor} \frac{(-1)^j}{w-j}\binom{w-j}{j} x^{w-2j}, \quad u_w(x) = \sum_{j=0}^{\lfloor (w-1)/2 \rfloor} (-1)^j \binom{w-1-j}{j} x^{w-1-2j}.$$

If w is half-integral,

$$t_w(x) = u_{w+1/2}(x) - u_{w-1/2}(x), \qquad u_w(x) = u_{w+1/2}(x) + u_{w-1/2}(x).$$

One should view the $T_w(x)$ and $U_w(x)$ as indexed by degree. Here boundary roots in $\{-2, 2\}$ count with multiplicity one half while interior roots in $(-2, 2)$ count with multiplicity one, in accordance with the presence of square roots. One has a variety of formulas connecting the Chebyshev polynomials, many direct translations of formulas more widely known in the context of cyclotomy and/or trigonometry. The connection between our Chebyshev polynomials and the most traditional ones is

$$t_w(x) = 2T_w^{\mathrm{tr}}(x/2), \qquad u_w(x) = U_{w-1}^{\mathrm{tr}}(x/2)$$

in the case of integral w. Note in particular the index shift in the case of Chebyshev polynomials of the second kind. Our notation places the focus on $U_w(x) = u_w^*(x)u_w(x)$ which does indeed have degree w. It also emphasizes primes of bad reduction as $\mathrm{disc}(u_w(x)) = 2^{w-1}w^{w-3}$.

3. Chebyshev covers

The Chebyshev polynomials having been defined, the definition of Chebyshev covers given in (1.1) is now complete. Basic facts about them can be established by direct computation. When possible, we treat the two cases simultaneously, writing F for either T or U throughout this paper.

First, we justify the conventions set up in the introduction by explaining how the excluded cases reduce to the considered cases. Suppose briefly that $m = m'd$ and $n = n'd$ for $d > 1$. Then

$$F_{m,n}(x) = F_{m',n'}(T_d(x))^d. \tag{3.1}$$

Similarly,

$$U_{m,n}(t^2, x) = -T_{m,n}(t, -x)T_{m,n}(-t, -x). \tag{3.2}$$

whenever m and n are both odd. Finally, and immediately from (1.1),

$$F_{m,n}(x) = F_{n,m}(x)^{-1}. \tag{3.3}$$

The standing assumption $m < n$ breaking the $m \leftrightarrow n$ symmetry is particularly convenient as some phenomena become associated to only m and others to only n. For example, in Section 6, vertices of valence related to n behave simply while those related to m behave in a complicated way. In Section 9, in reverse, primes dividing m behave more simply than primes dividing n.

The details of all our considerations depend on the parity of m and n. Often we must therefore break into five cases, naturally denoted $T01$, $T10$, $T11$, $U01$, and $U10$, the case $U11$ having been excluded. As an example of a case distinction, numerator and denominator in (1.1) are relatively prime in cases $T01$ and $T10$, but there is a cancellation in the remaining three cases.

The zeros of $T_{m,n}$ and $U_{m,n}$ from left to right have multiplicity

$$\begin{array}{lll} (m,n) \equiv (0,1): & \lambda^{T,0}_{m,n} = n^{m/2}, & \lambda^{U,0}_{m,n} = n, (2n)^{m/2-1}, k, \\ (1,0): & \lambda^{T,0}_{m,n} = \dfrac{n}{2}, n^{(m-1)/2}, & \lambda^{U,0}_{m,n} = (2n)^{(m-1)/2}, k. \\ (1,1): & \lambda^{T,0}_{m,n} = \dfrac{k}{2}, n^{(m-1)/2}, & \end{array} \tag{3.4}$$

The poles of $T_{m,n}$ and $U_{m,n}$ from left to right have multiplicity

$$\begin{array}{lll} (m,n) \equiv (0,1): & \lambda^{T,\infty}_{m,n} = \dfrac{m}{2}, m^{(n-1)/2}, & \lambda^{U,\infty}_{m,n} = (2m)^{(n-1)/2}, \\ (1,0): & \lambda^{T,\infty}_{m,n} = m^{n/2}, & \lambda^{U,\infty}_{m,n} = m, (2m)^{n/2-1}. \\ (1,1): & \lambda^{T,\infty}_{m,n} = m^{(n-1)/2}, & \end{array} \tag{3.5}$$

These zeros and poles are all in $[-2, 2]$ and so divide into interior singularities in $(-2, 2)$ and boundary singularities in $\{-2, 2\}$. Always interior zeros have multiplicity n in Case T and $2n$ in Case U. Always interior poles have multiplicity m in Case T and $2m$ in Case U. The case distinctions are important only for boundary singularities, of which there are always one in Case T and two in Case U. Our appropriation of the classical terminology of Chebyshev polynomials has an extra virtue not present in the classical theory: generally speaking, our functions $T_{m,n}$ and $U_{m,n}$ are similar; the main difference is that our functions of the second kind involve an extra factor of two in many ways.

While the left-to-right order in (3.4) and (3.5) is certainly of fundamental importance, often only $\lambda^{F,\sigma}_{m,n}$ as a partition of the degree enters into a given consideration. As partitions, $\lambda^{F,0}_{m,n}$ and $\lambda^{F,\infty}_{m,n}$ belong to a triple, the third member being the partition $\lambda^{F,1}_{m,n}$ giving the multiplicities of the preimages of 1. This third partition is $m1\cdots1$ except when k is a multiple of six, in which case it is $m21\cdots1$ as discussed below. Here m is the multiplicity of ∞ in the preimage of 1. It reflects

that $F_{m,n}(1,x)$ always has degree m less than $F_{m,n}(s,x)$, a fact which can be easily checked. In comparison to the other parts of our three partitions, the many 1's in $\lambda_{m,n}^{F,1}$ enter differently into our considerations. First, the corresponding roots are not critical points, exactly because their multiplicity is 1 rather than some larger number. Second, most of these roots are not real.

Throughout this paper we systematically illustrate general results with the particular cover

$$T_{8,9}(x) = \frac{\left(x^4 - 4x^2 + 2\right)^9}{(x-1)^8(x+2)^4\left(x^3 - 3x - 1\right)^8}.$$

Also we concentrate on the case $T01$ this cover represents. Figure 3.1 plots $T_{8,9}(x)$. The zeros on this plot interlace with the poles. This interlacing always occurs when

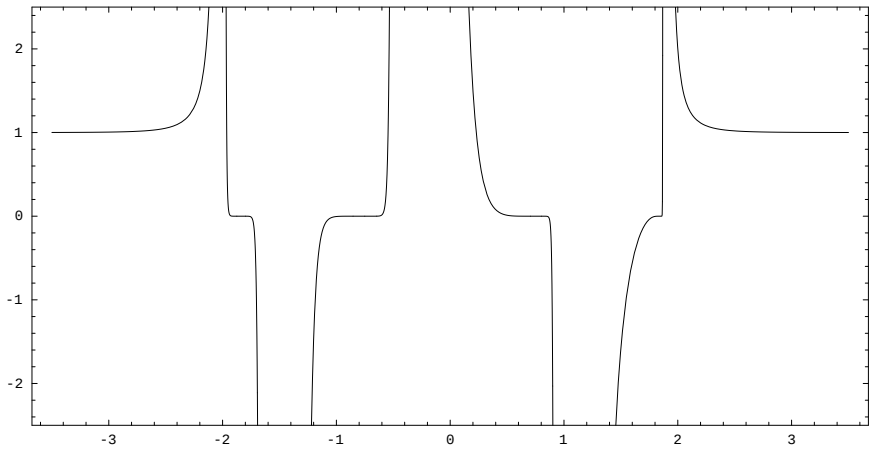

FIGURE 3.1. A plot of the rational function $T_{8,9}(x)$. Its four zeros of multiplicity nine and its five poles of high even multiplicity are clearly visible, as is the rapidly approached horizontal asymptote at $s = 1$.

$k = 1, 2$. It is usually not the case in general, as there are approximately $k/2$ more poles than zero. Rather the geometric situation is substantially more complicated because of the presence of approximately $k/2$ critical points, which we discuss next.

In Case $T01$, the $m/2$ zeros of multiplicity n each yield a contribution of $n-1$ to the critical divisor, for a total contribution of $(mn-m)/2$. The $(n-1)/2$ poles of multiplicity m and the further pole at $x = -2$ of multiplicity $m/2$ similarly yield a total contribution of $(mn-n-1)/2$ to the critical divisor. The point ∞ contributes $m-1$. The critical divisor of a rational function of degree N always has degree $2N-2$, which here is $mn-2$. This shows that there are $(n-m-1)/2 = (k-1)/2$ remaining critical points to be found. Similar simple computations for the other cases reveal that in general there are $\lfloor (k-1)/2 \rfloor$ remaining critical points to be found.

A first take on this situation is that the cases $k = 1$, 2 are worth pursuing while the cases $k > 2$ are not. Indeed we will focus on the cases $k = 1$, 2. However, arbitrary k is in fact a natural context. One argument for this is simply that our highest degree examples in Section 12 come from the setting $k = 3$. However a much more structural reason is given in Section 9: in the study of the bad reduction of a given $F_{m,n}$, other covers $F_{m',n}$ and $F_{m,n'}$ enter with very different index differences.

By taking the derivative of $F_{m,n}$ we find that these remaining critical points depend only on k, being always the roots of $u_{k/2}(x)$. We find that the corresponding critical values depend on whether one is in Case T or Case U, but again depend on m and n only through the difference $k = n - m$. Taking $m = 1$ and $n = k + 1$ to get the simplest formula, these critical values are the roots of the polynomials

$$\begin{aligned} d_k^T(s) &= \pm \mathrm{Res}_x \left((x+2)^{\lceil k/2 \rceil} - s t_{(k+1)/2}(x), u_{k/2}(x) \right), \\ d_k^U(s) &= \pm \mathrm{Res}_x \left((x-2)^k - s(x+2)^\delta u_{(k+1)/2}(x)^2, u_{k/2}(x) \right). \end{aligned}$$

Here δ either zero or one according to whether k is even or odd, and the sign is chosen so that $d_k^F(s)$ is always monic. The first few of these polynomials in factored form are

$$\begin{aligned} d_1^T(s) &= 1, & d_1^U(s) &= 1, \\ d_2^T(s) &= 1, & d_2^U(s) &= 1, \\ d_3^T(s) &= s+1, & d_3^U(s) &= s+27, \\ d_4^T(s) &= s+4, & d_4^U(s) &= s-16, \\ d_5^T(s) &= s^2+11s-1, & d_5^U(s) &= s^2+625s+3125, \\ d_6^T(s) &= (s-1)(s-27), & d_6^U(s) &= (s-1)(s-729). \end{aligned}$$

One has several recurring patterns among these polynomials, including in both cases that 1 is a root if and only if k is a multiple of 6. Thus usually one has $\lfloor (k-1)/2 \rfloor$ critical values beyond $\{0, 1, \infty\}$ but if k is a multiple of six, one has $\lfloor (k-1)/2 \rfloor - 1 = k/2 - 2$ extra critical values and the ramification partition for 1 takes the shape $m21^{N-m-2}$ rather than $m1^{N-m}$ as mentioned above.

4. Discriminant formulas

Our formulas for the discriminant of Chebyshev covers play a central role in our study so we state them here in all cases. The degree $N_{m,n}^F$ of $F_{m,n}$ enters repeatedly into our formulas. We write the degree as

$$\begin{aligned} &\text{Cases } T01,\, T10: \quad A = mn/2, && \text{Cases } U01,\, U10: \quad C = m(n-1). \\ &\text{Case } T11: \quad B = m(n-1)/2, \end{aligned}$$

to simplify the notation. Also we express some quantities in terms of the triangular numbers $\Delta(z) = z(z+1)/2$.

Theorem 4.1. *For $s \neq 1$, discriminants are as follows:*

$$T01: \quad disc(T_{m,n}(s,x)) = (-1)^{\Delta(m/2)+\Delta((n+1)/2)} 2^{A-m/2-n} m^A n^A (s-1)^{m-1} s^{A-m/2} d_k^T(s),$$

$$T10: \quad disc(T_{m,n}(s,x)) = (-1)^{(m-1)n/4} 2^{A-m-n/2} m^A n^A (s-1)^{m-1} s^{A-m/2-1/2} d_k^T(s),$$

$$T11: \quad disc(T_{m,n}(s,x)) = (-1)^{(m-1)(n-1)/4} m^B n^{B+(k-2)m-k} (s-1)^{m-1} s^{B-(m+1)/2} d_k^T(s),$$

$$U01: \quad disc(U_{m,n}(s,x)) = 2^C m^C n^{C+(2k-2)m-k} (s-1)^{m-1} s^{C-m/2} d_k^U(s),$$

$$U10: \quad disc(U_{m,n}(s,x)) = (-1)^{(m+n+1)/2} 2^{C-k} m^C n^{C+(2k-2)m-k} (s-1)^{m-1} s^{C-m/2-1/2} d_k^U(s).$$

The main new content here is the exponents on the arithmetic bases -1, 2, m, and n. The exponents on the main geometric factors s and $s-1$ and also the presence of the secondary geometric factor $d_k^F(s)$ were known from the previous section. We prove our discriminant formulas by methods similar to those used in [5], using again Equations (7.13)-(7.14) there as a starting point.

For $s = 1$, we indicate degree by $a = A - m$, $b = B - m$, and $c = C - m$. In the the same order of cases as before, we have the following complementary statements.

$$\begin{aligned}
\mathrm{disc}(T_{m,n}(1,x)) &= (-1)^{\Delta(m/2-1)+\Delta(n/2-3/2)} 2^{a-n/2-k/2} m^a n^{a-1} d_k^T(1),\\
\mathrm{disc}(T_{m,n}(1,x)) &= (-1)^{(m-1)(n+2)/4} 2^{a-n/2} m^a n^{a-1} d_k^T(1),\\
\mathrm{disc}(T_{m,n}(1,x)) &= (-1)^{(m-1)(n+1)/2} m^b n^{b+(k-2)m/2-k/2-1} d_k^T(1),\\
\mathrm{disc}(U_{m,n}(1,x)) &= (-1)^{m/2} 2^{c-1} m^c n^{c-1-k+(2k-2)m} d_k^U(1),\\
\mathrm{disc}(U_{m,n}(1,x)) &= (-1)^{n/2+1} 2^{c-1-k} m^c n^{c-1-k+(2k-2)m} d_k^U(1).
\end{aligned}$$

Similarly we have formulas for the discriminant of the separable part of $F_{m,n}(\sigma, x)$ when σ is a root of $d_{m,n}^F(s)$.

5. Dessins and monodromy

We sometimes use $-\infty$ as a synonym for the point ∞ in the base projective line $\mathbf{P}_s^1(\mathbf{C})$ when it seems more communicative. The geometric dessin $D_{m,n}^F$ of a cover $F_{m,n}$ is $F_{m,n}^{-1}([-\infty, 0])$ considered as a subset of $\mathbf{P}_x^1(\mathbf{C})$. Until the last paragraph we assume $k \leq 2$.

Figure 5.1 draws the geometric dessin $D_{8,9}^T$. The rightmost point of this dessin on the real line is the pole $2\cos(\pi/9) \approx 1.88$. The next rightmost point on the real line is the zero $2\cos(\pi/8) \approx 1.85$. These two parts are connected with eight edges.

Let $\Gamma_{8,9}^T$ be $D_{8,9}^T$ considered as combinatorial dessin. So $D_{8,9}^T$ is a specific subset of $\mathbf{C}$, while $\Gamma_{8,9}^T$ a planar graph which is allowed to be slid around freely. In

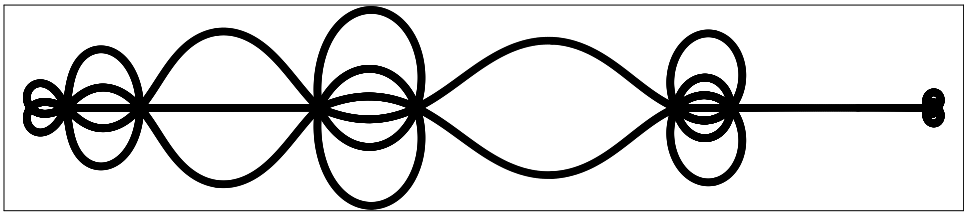

FIGURE 5.1. The dessin $T_{8,9}^{-1}([-\infty, 0])$ drawn in the region $[-2.1, 2.1] \times [-0.45, 0.45]$ of the complex x-plane. The five poles of are interspersed with four zeros connected by edges in accordance with the diagram (5.1). The roots of $T_{8,9}(-1, x)$ mark the centers of the thirty-six edges while the roots of $T_{8,9}(1, x)$ mark the centers of the twenty-eight bounded faces.

practice, geometric dessins are computer drawings, while combinatoric dessins are more truly children's drawings. The distinction is fundamental, as indeed one often views combinatorial dessins as the input to the theory and geometric dessins as the output. Nonetheless, we normally say simply dessin, as the context is clear.

Besides geometric dessins D and combinatorial dessins Γ, a closely related third object γ comes into play. We call this third object the reduced combinatorial dessin, or again just dessin. It is constructed from Γ by iteratively identifying two edges which together bound a face, losing also the bounded face in the process. The lost face corresponds to a non-critical point in the fiber $F^{-1}(1)$. Collapsing edges two at a time in this way, many edges can be collapsed to one, and γ is to be viewed as a bipartite weighted planar graph. The weight of a vertex of γ is the valence of that vertex in Γ. The weight on an edge of γ is the number of edges in Γ reducing to it. So edge weights determine vertex weights. However one often keeps the focus on vertex weights since they are the more basic quantities.

In our example, the reduced combinatoric dessin $\gamma_{8,9}^T$ is

$$\gamma_{8,9}^T = \mathbf{4} \overset{4}{-} 9 \overset{5}{-} \mathbf{8} \overset{3}{-} 9 \overset{6}{-} \mathbf{8} \overset{2}{-} 9 \overset{7}{-} \mathbf{8} \overset{1}{-} 9 \overset{8}{-} \mathbf{8}. \tag{5.1}$$

Here and in the sequel, numbers in bold are multiplicities of poles, while numbers in regular type are multiplicities of zeros.

Since the partition $\lambda_{m,n}^{F,1}$ controlling faces is $m1 \cdots 1$, it is clear from the definition that the graph $\gamma_{m,n}^F$ is a tree in general. We can prove that, as one might expect from the example (5.1), that $\gamma_{m,n}^F$ is in fact always a segment. The proof is elementary, using the fact that all vertices are real.

As always for dessins, the monodromy group is generated by operators g_0 and g_∞ acting on the edges of Γ by rotating a given edge minimally counterclockwise about the endpoint which is a zero or pole respectively. The highly structured nature of our dessins lets us label edges of Γ in simple ways so that the monodromy action can be written down in algebraic terms.

The dessin $D^F_{m,n}$ brings to visual prominence two polynomials of particular interest. First, it is reasonable to call the N roots of $F_{m,n}(-1,x)$ the centers of the N edges. Similarly, it is reasonable to view the $N-m$ roots of $F_{m,n}(1,x)$ as the centers of the bounded faces. In particular, we have a well-defined way to label roots, even though the analytic fact that they lie approximately in columns is not confirmed. We use this labeling in Section 10.

One can think of dessins more dynamically, viewing s as representing time and the dessin as traced out by N moving points, distinguishable in the interval $(-\infty, 0)$. In our cases, the particles start out at time $s=-\infty$ clumped at the approximately $n/2$ poles. They expand in circles of generic size m or $2m$ about the fixed poles until approximately $s=-1$ where circles about boundary poles have moved inward to approximately vertical lines while circles about interior poles have split into two approximately vertical columns. Then interior columns pair with their other adjacent column, and the process reverses as the points contract in circles of generic size n or $2n$ to the approximately $m/2$ zeros at $s=0$.

For general k, the dynamic description just given goes through in large part. The difference is that a pair of real roots can coalesce to become a pair of conjugate non-real roots and one has to make choices to label roots consistently over all of $(-\infty, 0)$. The extra coalescence is necessary for a consistent picture to account for the approximately $n/2$ initial circles becoming approximately only $m/2$ circles.

6. QuasiChebyshev covers

In this section, we restrict to the cases $k=1,2$ and thus three point covers. We define a three point cover to be a quasiChebyshev cover if its ramification partitions over 0, 1, and ∞ agree with a Chebyshev cover and if it's normalized in the same way as a Chebyshev cover, as we'll explain. The set $\mathcal{F}_{m,n}$ of quasiChebyshev covers agreeing numerically with a given cover $F_{m,n}$ is finite and can be computed by solving equations.

In Case $T01$, we begin our normalization by requiring that a quasiChebyshev cover send ∞ to 1 and -2 to ∞ just as $T_{m,n}=T_{m,m+1}$ itself does. Then we have the general form

$$T^{\mathrm{gen}}_{m,n}(x) \quad = \quad \frac{\left(x^{m/2}+\sum_{i=1}^{m/2} a_i x^{m/2-i}\right)^n}{(x+2)^{m/2}\left(x^{m/2}+\sum_{i=1}^{m/2} c_i x^{m/2-i}\right)^m}.$$

We look at the "new factor" $\Delta^T_{m,n}(x)$ of the numerator of the derivative of $T^{\mathrm{gen}}_{m,n}(x)$. The rational functions we seek are those for which $\Delta^T_{m,n}(x)$ is reduced to a constant.

An affine transformation fixes -2 if and only if it has the form $x\mapsto \lambda x+(2\lambda-2)$. If we have a solution then changing x by this affine transformation gives another solution. There are two cases to distinguish. The highest term of $\Delta^T_{m,n}(x)$ is $((m+1)a_1-m(c_1+1))x^m$. Either a_1 and c_1 are both m, or they are both different from m. In the former case, the three point covers we construct have non-trivial automorphisms while in the latter case they do not. We focus on the latter case first, which is the main case. In this main case, we complete our normalization by requiring $a_1=0$ or equivalently $c_1=-1$.

In our continuing example, we have

$$\begin{aligned}\Delta_{8,9}^T(x) &= (-24-18a_2+16c_2)x^6 \\ &\quad (-18a_2-27a_3+40c_2+24c_3)x^5 + \\ &\quad (12a_2-27a_3-36a_4-2a_2c_2+56c_3+32c_4)x^4 + \\ &\quad (30a_3-36a_4+4a_2c_2-11a_3c_2+6a_2c_3+72c_4)x^3 + \\ &\quad (48a_4-14a_3c_2-20a_4c_2+20a_2c_3-3a_3c_3+14a_2c_4)x^2 + \\ &\quad (-32a_4c_2+2a_3c_3-12a_4c_3+36a_2c_4+5a_3c_4)x + \\ &\quad (-16a_4c_3+18a_3c_4-4a_4c_4).\end{aligned}$$

Equating the coefficients of x^6, x^5, x^4, and x^3 successively to zero gives

$$\begin{aligned}c_2 &= (12+9a_2)/8, \\ c_3 &= (-20-9a_2+9a_3)/8, \\ c_4 &= (560+216a_2+9a_2^2-144a_3+144a_4)/128, \\ a_4 &= (-112-40a_2-a_2^2+24a_3+2a_2a_3)/16.\end{aligned}$$

Writing $a=a_2$ and $b=a_3$ we then have

$$\begin{aligned}&\frac{256}{9}\Delta_{8,9}^T(x) = \\ &\quad 8\left(5a^3-3ba^2+60a^2-48ba+48a-12b^2+48b-448\right)x^2 + \\ &\quad 4\left(10a^3+22ba^2+120a^2-7b^2a+72ba+544a-108b^2+320b+896\right)x + \\ &\quad \left(ba^3-40a^3-2b^2a^2+156ba^2-1696a^2-24b^2a+2128ba-8320a-\right. \\ &\quad \left.576b^2+4544b-10752\right).\end{aligned} \tag{6.1}$$

The variable b occurs quadratically, so we can not eliminate it by such elementary algebra. Instead, we take the resultant of the coefficients $h_2(a,b)$ and $h_1(a,b)$ of x^2 and x respectively in (6.1) to get a constant times

$$\begin{aligned}g_{8,9}^T(a) &= (a+4)\left(35a^7+2380a^6+38192a^5+236480a^4+928000a^3+\right. \\ &\quad \left.3015680a^2-3993600a-16564224\right).\end{aligned} \tag{6.2}$$

The septic polynomial on the right has Galois group S_7 and field discriminant $-2^4 3^5 5^6 7^2 11^5 19^3$. Each root α of $g_{8,9}^T(a)$ determines a quasiChebyshev cover $T_{m,n}^{[\alpha]}(x)$ with $\alpha=-4$ yielding the Chebyshev cover $T_{8,9}(x)$.

In general, the part of $\mathcal{T}_{m,m+1}$ consisting of covers without extra automorphisms is likewise indexed by the roots of a suitable moduli polynomial $g_{m,m+1}^T(a)$. The rest of $\mathcal{T}_{m,m+1}$, as we'll see topologically, is indexed by divisors d of m besides 1 and roots a of another moduli polynomial $g_{m,n}^{T,[d]}$ so that $T_{m,n}^{[d;\alpha]}$ has exactly d automorphisms. The cases $T10$, $T11$, $U01$, and $U11$ are easier in that there are no quasiChebyshev polynomials with extra automorphisms.

We have computed all cases up through $|\mathcal{F}_{m,n}|=42$ as listed on Table 7.1. In this range, the polynomial $g_{m,n}^F(a)$ always factors over $\mathbf{Q}$ into a linear factor corresponding to the Chebyshev cover $F_{m,n}$ and a complementary irreducible factor corresponding to the other quasiChebyshev covers without extra automorphisms, just like in (6.2). Always the Galois group of the complementary factor is the full symmetric group on its degree. Always, except for very low degrees, the moduli polynomial is ramified at primes beyond those dividing m and n. For

example, the field discriminant of the degree thirty four polynomial for $U_{8,9}^{\text{gen}}$ is $2^{71}3^{44}5^{27}7^{27}11^{23}13^{19}19^{15}23^{10}29^{11}31^{8}37^{4}47^{3}$. The three point covers themselves can be further ramified beyond the ramification in the moduli polynomial. For example, $T_{8,9}^{[2;]}$, with equation given in Figure 7.1, is defined over $\mathbf{Q}$ but has bad reduction at 5 and 7 as well as at 2 and 3. In short, in the collection of quasiChebyshev covers only the Chebyshev covers seem to be arithmetically special.

One could go further in relating our covers to other covers as follows. One can demand that $\Delta_{m,n}^{T}$ be simply linear, rather than constant, thereby generically seeking covers with ramification partitions $(\lambda_{m,n}^{0}, (m-1)1\cdots 1, \lambda_{m,n}^{\infty})$ above $(0, 1, \infty)$ and a fourth unspecified ordinary ramification point. The solution set $\mathcal{F}_{m,n}^{1}$ is a curve containing $\mathcal{F}_{m,n}$. Thus this approach embeds the finite set $\mathcal{F}_{m,n}$ in a single connected family. In our example case, $\mathcal{T}_{8,9}^{1}$ is the elliptic curve of conductor $1210 = 2 \cdot 5 \cdot 11^2$ defined by $h_2(a, b) = 0$.

We have not yet used the constant term $h_0(a, b)$ of $\Delta_{8,9}^{T}(x)$. However, using the same notation to indicate the general case, the root $x_{\text{crit}} = -h_0(a, b)/h_1(a, b)$ of the linear polynomial $\Delta_{m,n}^{F}(x)$ can be viewed as a function on the curve $\mathcal{F}_{m,n}^{1}$. Its critical value $s_{\text{crit}} = F_{m,n}^{\text{gen}}(x_{\text{crit}})$ can likewise be viewed as a function on $\mathcal{F}_{m,n}^{1}$. In fact, the function s_{crit} presents $\mathcal{F}_{m,n}^{1}$ as a three point cover of the line with coordinate s_{crit}. Our set $\mathcal{F}_{m,n}$ is in the fiber above ∞. The rest of the fiber above ∞ and the entire fibers above 0 and 1 include other sets analogous to $\mathcal{F}_{m,n}$, indexing three point covers of different partition triples. Already our example case $T_{8,9}$ is complicated, but lower degree cases which satisfy $\mathcal{F}_{m,n}^{1} \cong \mathbf{P}^1$ are computationally easy. Even in this enlarged context, the Chebyshev covers seem to be the only three point covers which are arithmetically special.

7. QuasiChebyshev Dessins

In this section, we again restrict to the cases $k = 1, 2$. We explain how "half" of the topological simplicity of Chebyshev covers is kept by quasiChebyshev covers. This allows us to index the sets $\mathcal{F}_{m,n}$ of quasiChebyshev covers in a particularly simple way.

A quasiChebyshev cover has a dessin, again simply the preimage D of $[-\infty, 0]$. It has a combinatoric dessin Γ and a reduced combinatoric dessin γ, respectively a planar bipartite graph and a planar bipartite weighted tree. At issue is to say explicitly what the possibilities for γ are.

The vertex weights on γ are given, being by definition exactly the same as in the Chebyshev case. The sum of the weights of edges incident on a vertex is exactly the given vertex weight. Vertex weights then completely determine edge weights, but many candidates for γ yield zero or negative edge weights. For example, the weighted bipartite tree

$$\gamma = \begin{array}{ccccccccccccc} \mathbf{8} & \overset{8}{-} & 9 & \overset{1}{-} & \mathbf{4} & \overset{3}{-} & 9 & \overset{6}{-} & \mathbf{8} & \overset{2}{-} & 9 & \overset{8}{-} & \mathbf{8} \\ & & & & & & & & & & \vert{\scriptstyle -1} & & \\ & & & & & & & & & & \mathbf{8} & & \\ & & & & & & & & & & \vert 9 & & \\ & & & & & & & & & & 9 & & \end{array}$$

is not a reduced combinatorial dessin because of the negative edge weight -1. Clearly for the partitions $\lambda_\infty = 48^4$ and $\lambda_0 = 9^4$ in this example, a weight nine

vertex can never have valence one. It then follows that it can never have valence ≥ 3 either, and thus must have valence two.

In general, we say a zero-vertex is large if it has weight n in Case T or $2n$ in Case U. Similarly, a polar-vertex is large if has weight m in Case T or $2m$ in Case U. Otherwise vertices are medium or small, meaning half the generic weight or weight 1 respectively. There is one small or medium vertex in Case T and two small or medium vertices in Case U, as mentioned in Section 3. The numerics force in a very simple way, illustrated for $T_{8,9}^{\text{gen}}$ above, that the large zero-vertices have valence two in all cases $T01$, $T10$, $T11$, $U01$, and $U10$ and the medium or small zero-vertices have valence one. More surprisingly, in all cases, there is no condition on the polar vertices, whether they be large, medium, or small. For example, the weighted bipartite trees

$$\mathbf{8}\overset{8}{-}9\overset{1}{-}\mathbf{8}\overset{7}{-}9\overset{2}{-}\begin{array}{c}\mathbf{8}\\ |5\\ 9\\ |4\\ \mathbf{4}\end{array}\overset{1}{-}9\overset{8}{-}\mathbf{8} \qquad\qquad \mathbf{8}\overset{8}{-}9\overset{1}{-}\mathbf{8}\overset{7}{-}9\overset{2}{-}\begin{array}{c}\mathbf{4}\\ |1\\ 9\\ |8\\ \mathbf{8}\end{array}\overset{1}{-}9\overset{8}{-}\mathbf{8}$$

are each reduced combinatorial dessins. They each have polar vertices of valence 1, 2, and 3.

Because edge weights are automatic from vertex weights, we don't need to write them. Because all zero-vertices have valence one or two, we only need to draw in the former. We distinguish between small and medium zero-vertices by drawing nothing and $\star$ respectively. This last distinction is important only in Case $U01$, since only in this case is there both a small and medium zero-vertex. By this procedure, any reduced combinatorial dessin γ gives rise to a combinatorial object δ which we call a polar combinatorial dessin. The reduction $\gamma \mapsto \delta$ is bijective, and we focus on the δ's.

$$\delta = \mathbf{8}-\mathbf{8}-\mathbf{4}-\mathbf{8}-\mathbf{8} \qquad T_{8,9}^{\delta}(x) = \frac{\left(z^2-24z+84\right)^9}{z^2\left(z^2-27z+135\right)^8} \quad \text{with } z=\tfrac{5}{4}(x+2)^2$$

$$\delta = \begin{array}{c}\mathbf{8}\\ |\\ \mathbf{8}-\mathbf{4}-\mathbf{8}\\ |\\ \mathbf{8}\end{array} \qquad T_{8,9}^{\delta}(x) = \frac{(z-8)^9}{z(z-9)^8} \quad \text{with } z=9(x+2)^4$$

FIGURE 7.1. The two polar dessins δ indexing quasiChebyshev dessins in $\mathcal{T}_{8,9}$ with rotational symmetry, and for each a formula for $T_{8,9}^{\delta}$.

We now systematically use the letter e to index objects. We use polar dessins δ to distinguish quasiChebyshev covers from each other. Thus, in Case $T01$, the elements of $\mathcal{T}_{2e,2e+1}$ are $T^{\delta}_{2e,2e+1}$ where δ runs over planar trees with a marked vertex. In our continuing example $T^{\mathrm{gen}}_{8,9}$, the possibilities for δ are given in Figures 7.2 and 7.1.

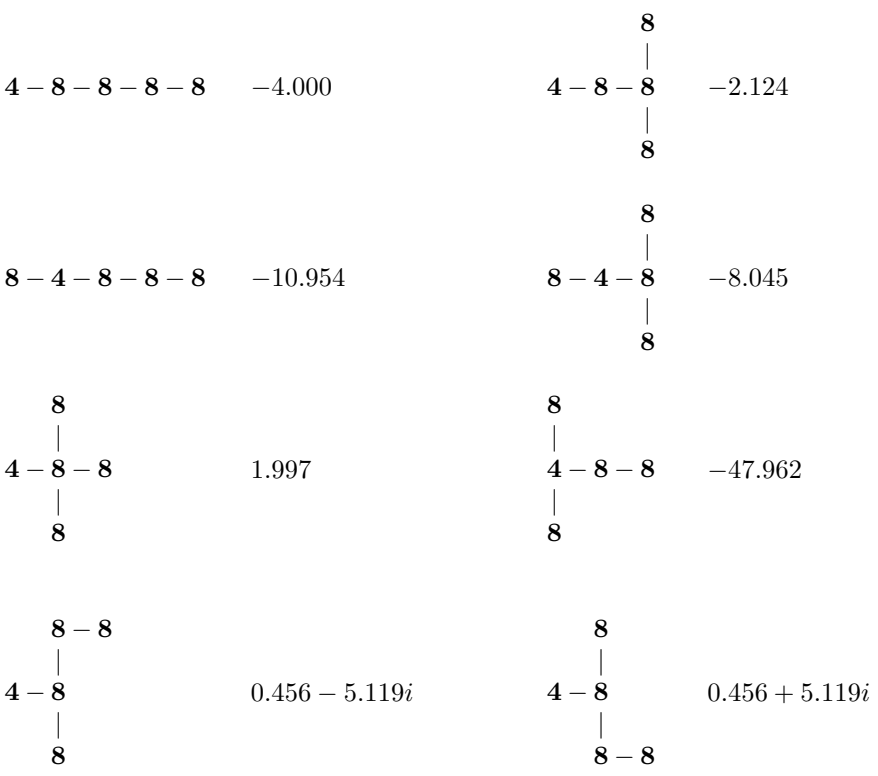

FIGURE 7.2. The polar dessins δ indexing quasiChebyshev dessins in $\mathcal{T}_{m,n}$ without rotational symmetry and for each the corresponding root of (6.2).

Our terminology "polar dessin" is self-explanatory in Case $T01$ as δ refers to poles only, not zeros. In the remaining cases, the term reflects the fact that δ refers to all the poles, and only the small and medium zeros, of which there is at most one of each type. Figure 7.3 illustrates our drawing conventions for polar dessins in each case.

Recall that the e^{th} Catalan number is

$$C_e = \frac{1}{e+1}\binom{2e}{e} = \frac{(2e)!}{(e+1)!e!}. \tag{7.1}$$

For $e = 1, \ldots, 5$, the corresponding Catalan numbers are 1, 2, 5, 14, 42 and asymptotically one has $C_e \sim 4^e/(\sqrt{\pi}e^{3/2})$. The usual statement is that Catalan numbers count rooted planar trees, meaning a planar tree together with a marked

$$\gamma^T_{6,7} = \mathbf{3} - 7 - \mathbf{6} - 7 - \mathbf{6} - 7 - \mathbf{6} - 7 \qquad \delta^T_{6,7} = \mathbf{3} - \mathbf{6} - \mathbf{6} - \mathbf{6}$$

$$\gamma^T_{5,6} = 3 - \mathbf{5} - 6 - \mathbf{5} - 6 - \mathbf{5} \qquad \delta^T_{5,6} = \star\text{-}\,\mathbf{5} - \mathbf{5} - \mathbf{5}$$

$$\gamma^T_{5,7} = 1 - \mathbf{5} - 7 - \mathbf{5} - 7 - \mathbf{5} \qquad \delta^T_{5,7} = \text{-}\,\mathbf{5} - \mathbf{5} - \mathbf{5}$$

$$\gamma^U_{6,7} = 7 - \mathbf{12} - 14 - \mathbf{12} - 14 - \mathbf{12} - 1 \qquad \delta^U_{6,7} = \star\text{-}\mathbf{12} - \mathbf{12} - \mathbf{12}\text{-}$$

$$\gamma^U_{5,6} = \mathbf{5} - 12 - \mathbf{10} - 12 - \mathbf{10} - 1 \qquad \delta^U_{5,6} = \mathbf{5} - \mathbf{10} - \mathbf{10}\text{-}$$

FIGURE 7.3. On the left, the reduced combinatoric dessin of Chebyshev covers representing each of the five cases. On the right, the corresponding polar dessins. The procedure for passing from a reduced combinatoric dessin γ to the corresponding polar dessin δ does not use the linear structure present in these diagrams, and works in the quasiChebyshev context.

Case	Description	Mass	Masses for $e = 1, 2, 3, \ldots$
$\mathcal{T}_{2e,2e+1}$	Vertex marked as medium	$\frac{C_e}{2e}(e+1)$	$.\overline{6}, 1.5, 3.\overline{3}, 8.75, 25.2, \ldots$
$\mathcal{T}_{2e+1,2e+2}$	Medium half-edge	C_e	$1, 2, 5, 14, 42, \ldots$
$\mathcal{T}_{2e+1,2e+3}$	Small half-edge	C_e	$1, 2, 5, 14, 42, \ldots$
$\mathcal{U}_{2e+2,2e+3}$	Medium half-edge Small half-edge	$C_e(2e+1)$	$3, 10, 35, \ldots,$
$\mathcal{U}_{2e+1,2e+2}$	Vertex marked as medium Small half-edge	$C_e(e+1)$	$2, 6, 20, \ldots$

TABLE 7.1. Description and masses of the sets $\mathcal{F}_{m,n}$. In each case the description is in terms of what needs to be added to a planar tree with e edges to get a polar dessin δ. The polar dessins δ can have non-trivial rotational symmetry only in the first case $T01$. Otherwise, masses agree with cardinalities.

vertex and a marked edge incident upon it. Another point of view is that the rational numbers $C_e/(2e)$ give the mass of planar trees with e edges, the mass of a planar tree τ being as usual $1/|\mathrm{Aut}(\tau)|$ with $\mathrm{Aut}(\tau)$ its group of symmetries. Our polar dessins with e edges are constructed from planar trees with e edges by distinguishing vertices and/or adding half-edges. Marking a vertex corresponds to multiplying by the number of vertices $e+1$. Adding a half edge corresponds to multiplying by $2e$. Adjoining a second half edge corresponds to multiplying by $2e+1$. The total mass of the sets $\mathcal{F}_{m,n}$ is thus as given in Table 7.1.

8. Generic Monodromy

Recall our standing conventions that m and n are relatively prime and not both odd in the Case U. Recall also that quasiChebyshev covers can have non-trivial automorphisms only in Case $T01$. The main result of this section is as follows.

Theorem 8.1. *A quasiChebyshev cover without non-trivial automorphisms has monodromy group the full alternating or symmetric group on its degree.*

Our proof of this monodromy result proceeds in stages as follows. Irreducibility is obvious from the fact that $\mathbf{P}^1_x$ is connected. Primitivity follows from the fact that the ramification partitions associated to $F_{m,n} : \mathbf{P}^1_x(\mathbf{C}) \to \mathbf{P}^1_s(\mathbf{C})$ do not allow for an intermediate curve, unless $F_{m,n}$ has automorphisms. Finally irreducibility is deduced from the fact that the monodromy group contains g_1 which has cycle structure $m1^{N-m}$. The fact that the proof goes through in the quasi setting indicates the naturality of this setting.

9. Field discriminants of specializations

One would ideally like to have explicit descriptions of p-adic ramification in the polynomials $F_{m,n}(s,x)$ for all $F_{m,n}$, all primes p, and all $s \in \mathbf{Q}_p^\times$. Experimentation shows very regular behavior in all cases.

Let $\overline{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$. Let $\mathbf{Q}_p^{\mathrm{un}}$ be the induced maximal unramified extension of $\mathbf{Q}_p$. It is best to work geometrically, meaning factoring over $\mathbf{Q}_p^{\mathrm{un}}$ rather than $\mathbf{Q}_p$. The dominant phenomenon is that $F_{m,n}(s,x)$ factors p-adically into factors which look very similar to each other with a few exceptions.

A clean example is provided by $T_{8,9}(-1,x)$ with $p=3$. Its polynomial discriminant at 3 is 3^C with $C=72$. It factors over $\mathbf{F}_3$ as $x^9(x+1)^9(x^2+1)^9$ and thus over $\overline{\mathbf{F}}_3$ as $x^9(x+1)^9(x+i)^9(x-i)^9$. Write the field discriminant at 3 as 3^c. Then both C and c must be distributed somehow over the four roots, via $C=\sum C_r$ and $c=\sum c_r$. Necessarily $C_r - c_r$ is non-negative and even. The simplest behavior that one could hope for is even distribution and no drop, so that the C_r and c_r are all $72/4=18$. This is indeed what happens.

A more representative example is provided by $T_{8,9}(-1,x)$ with $p=2$. It factors over $\mathbf{F}_2$ as $x^4(x^3+x^2+1)^8$, giving roots 0, r_1, r_2, r_3 over $\overline{\mathbf{F}}_2$. Also, because of the degree drop, ∞ must be considered a root of multiplicity eight. Discriminant exponents (C_r, c_r) are $(35,11)$ for the quartic root $x=0$, $(24,24)$ for the octic factors corresponding to the r_i, and $(31,31)$ for $x=\infty$. Here one should regard the r_i as behaving typically and 0 and ∞ as both behaving specially.

The starting point for analysis in general is a factorization modulo p, as follows. If p^j exactly divides n, define

$$e(T,m,n,p) = \begin{cases} \min(n/2, m2^{j-1}) & \text{if } p=2, \\ \min(k/2, m(p^j-1)/2) & \text{if } p>2, \end{cases}$$

$$e(U,m,n,p) = \min(k, m(p^j-1)).$$

Then for p^j exactly dividing m or n as indicated and for s reduced to $\mathbf{F}_p \cup \{\infty\}$, one has congruences

$$\begin{aligned} T_{m,n}(s,x) &\equiv T_{m/p^j,n}(s,x)^{p^j}(x+2)^{m/2}, \\ T_{m,n}(s,x) &\equiv T_{m,n/p^j}(s,x)^{p^j}(x+2)^{e(T,m,n,p)}, \\ U_{m,n}(s,x) &\equiv U_{m/p^j,n}(s,x)^{p^j}, \\ U_{m,n}(s,x) &\equiv U_{m,n/p^j}(s,x)^{p^j}(x-2)^{e(U,m,n,p)}, \end{aligned}$$

modulo p.

When s reduces into $\mathbf{F}_p^\times$ rather than into $\{0,\infty\}$, the factorizations just displayed are particularly powerful. For then the bases on the right are generically separable, having at worst a double root when s is a root of the relevant $d^F_{m',n'}$. The factor $f(x)$ of $F_{m,n}(s,x)$ over $\mathbf{Q}_p^{\mathrm{un}}$ corresponding to a generic root is irreducible of degree p^j. It has polynomial discriminant p^{jp^j} and field discriminant also p^{jp^j}. Moreover $K_j = \mathbf{Q}_p^{\mathrm{un}}[x]/f(x)$ has an increasing chain of subfields K_i of degree p^i and field discriminant p^{jp^i}. Thus the slopes measuring wild ramification in the normalization of [2] are $i + 1/(p-1)$ for $i = 1, \ldots, j$. This is less than the maximally wild case for extensions of $\mathbf{Q}_p^{\mathrm{un}}$ of degree p, as there slopes are $i + 1 + 1/(p-1)$ for $i = 1, \ldots, j$. The source of exceptional factors are the special points $x = -2$, 2, ∞, and the roots of the relevant $U_{k'/2}$. Behavior of these factors needs to be described in a case-by-case way and often involves tame ramification.

When s reduces into $\{0,\infty\}$, the p-adic factorization may involve larger degree factors. For example, $T_{8,9}(2,x)$ has an irreducible degree 32 factor over $\mathbf{Q}_2^{\mathrm{un}}$. Also slopes of generic factors can reach the maximum of $j+1+1/(p-1)$. For example, $T_{8,9}(3,x)$ factors over $\mathbf{Q}_3^{\mathrm{un}}$ into four nonics, each having the maximum possible discriminant 3^{26}.

10. Imprimitive Specializations

Consider a Chebyshev cover, thought of as a family of polynomials $F_{m,n}(s,x)$ with generic Galois group G_N either A_N or S_N. We know that for generic $\sigma \in \mathbf{Q}$, the Galois group of $F_{m,n}(\sigma,x)$ is all of G_N. However we are interested in constructing number fields by specializing at the "most special" points σ. So there is some concern that Galois groups will drop dramatically at these points.

In this section, we experimentally find that there is indeed such a drop in two instances. First, for $k = n - m$ odd, $U_{m,n}(1,x)$ has degree $m(n-2)$. Within range of computation it always is irreducible, but it has subfield of index two. Moreover, this subfield is defined by $T_{m,n}(1,x)$ which has generic Galois group. Second, for k arbitrary now, $T_{m,n}((-1)^k,x)$ has either 0, 1, or 2 factors of $(x+1)$ and the remaining part $T^*_{m,n}((-1)^k,x)$ has degree a multiple of three. Within range of computation it is always irreducible, but has a subfield of index three.

For $k = 1, 2$ we strengthen the obvious conjecture that this pattern continues by being explicit as to how roots are to be grouped. The second case concerns $T_{m,m+1}(-1,x)$ for general m and $T_{m,m+2}(1,x)$ for m odd. The parity distinction drops out, and counting -1 only once even if it has multiplicity two, there are always exactly $m(m-1)/2$ roots. The roots, as suggested by Figure 5.1 in the case $T_{8,9}$ form columns of heights 1 through $m-1$ in the case $k = 1$ and 0 through $m-1$ in the case $k = 2$. For a, b, c positive integers summing to $m+1$, let α_{abc} be

the b^{th} root from the bottom or top in the column with a roots; here one counts alternately from the bottom or top as one considers the columns from left to right. Define new complex numbers $\beta_{abc} = \alpha_{abc} + \alpha_{cab} + \alpha_{bca}$, excluding the central case $a = b = c$ if it is present. Then the conjecture is that the monic polynomial with roots β_{abc} is in fact in $\mathbf{Q}[x]$. Our triangular indexing on the roots of $T_{8,9}(-1, x)$ is indicated in Figure 10.1. The corresponding degree twelve polynomial

$$f(x) = x^{12} - 36x^{10} - 48x^9 + 378x^8 + 864x^7 - 984x^6 - 4320x^5 - 3285x^4 + 192x^3 + 864$$

indeed has Galois group S_{12}. Our strengthening of the 2-imprimitivity conjecture likewise makes of triangular indices for the roots of $T_{m,n}(1, x)$ and two sets of triangular indices for the roots of $U_{m,n}(1, x)$. It is to be hoped that a proof of our imprimitivity conjectures would add insight to the nature of Chebyshev covers.

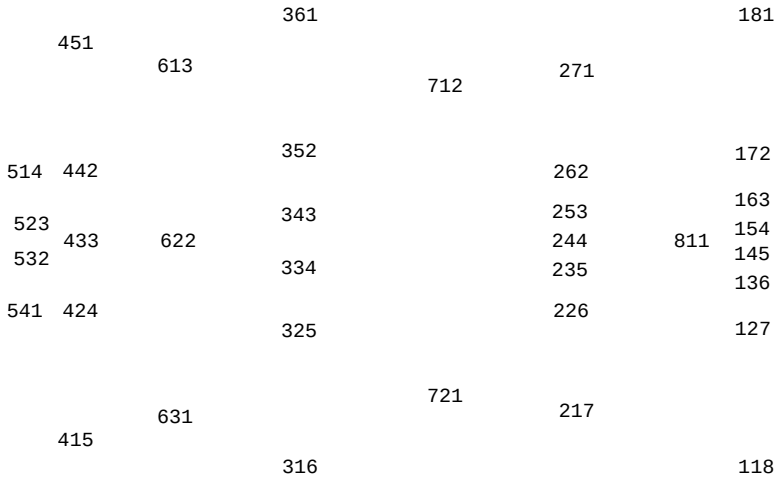

FIGURE 10.1. Triangular labels on the roots of $T_{8,9}(-1, x)$. Each label is placed at the corresponding root, except that imaginary parts are independently scaled in each column for better visibility. Root α_{abc} has $9 - a$ roots in its column, and is either b^{th} from the top and c^{th} from the bottom, or vice versa, depending on the parity of the column.

Besides the mysterious imprimitivity phenomenon, there are two obvious sources of Galois drop. First, if σ is of the form $F_{m,n}(x_0)$ then certainly $F_{m,n}(\sigma, x)$ has x_0 as a root and so $G_\sigma \subseteq G_{N-1}$. Second, if the discriminant $D_{m,n}(s)$ is not a square in $\mathbf{Z}[s]$ but $D_{m,n}(\sigma)$ is a square in $\mathbf{Z}$ then certainly $G_\sigma \subseteq A_N$ while the generic Galois group is S_N. For very low (m, n), there are other systematic sources of Galois drops because the curves governing the drop to a given group may have genus zero. For example, consider the quartic cover family $U_{2,3}(s, x) = (x - 2)(x + 2)^3 - s(x + 1)^4$ with Galois group S_4. It has $T_{2,3}(s, y)$ as its resolvent cubic. So for σ of the form $T_{2,3}(y) = y^3/((y - 1)^2(y + 2))$, the quartic $U_{2,3}(\sigma, x)$ has Galois group within the

dihedral group D_4. In all but very small degrees, there seem to be no further Galois drops. This lack of Galois drops is what we want for the purposes of the next two sections.

11. Moderate degree exceptional fields with discriminant $\pm 2^a 3^b$ or $\pm 3^a 5^b$

Consider degree N number fields with Galois group A_N or S_N and discriminant divisible only by primes in a given finite non-empty set S. In [7], the expected number of such fields was discussed and in particular such a field was defined to be *exceptional* when N is larger than a certain number $N(S)$. There is currently no general way to construct exceptional fields for any given S, and it was conjectured in [7] that for each S there are only finitely many exceptional fields.

Specializing Chebyshev covers gives exceptional fields for many S. Moreover, the fields go substantially beyond the already very stringent demand $N > N(S)$ in two ways. First, very simply, N can be very much greater than $N(S)$. But second, as described in Section 9, the fields constructed here are ramified much more lightly than is allowed by their degree.

Jordan's criterion. Jordan's criterion says that a transitive subgroup of S_N containing an element or prime order $\ell \in (N/2, N-3]$ must be A_N or S_N. To confirm the genericity of a generic polynomial $f(x) \in \mathbf{Z}[x]$ using this criterion, one verifies that $f(x)$ is irreducible over $\mathbf{Z}$ and finds a prime p such that $f(x) \in \mathbf{F}_p[x]$ has an irreducible factor of degree ℓ. Using say *Mathematica 6* this is very easy in practice for degrees ≤ 1000 and we will not generally further comment on verifications of genericity.

Fields with $S = \{2, 3\}$. The case $S = \{2, 3\}$ has been specifically pursued in the literature. The main previous paper is [5], where fields of degree up through 33 are constructed. The main technique in [5] is specializing three and four point covers, exactly as in this paper.

One has $N(\{2, 3\}) = 62$ and so the previous fields do not come close to being exceptional. Our family $T_{8,9}$ with generic degree 36 goes slightly beyond the previous 33 while our family $U_{8,9}$ with generic degree 64 goes slightly into the exceptional range.

Of course, it remains to confirm that specializations of $T_{8,9}$ and $U_{8,9}$ behave generically. As in [5], besides $s = 1$ we use the twenty-one specialization points

$$\left\{-8, -3, -2, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{8}, \frac{1}{9}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{8}{9}, \frac{9}{8}, \frac{4}{3}, \frac{3}{2}, 2, 3, 4, 9\right\} \tag{11.1}$$

coming from the orbits of 2, 3, 4, and 9 under the action of S_3 permuting the three cusps $s = 0$, 1, ∞.

As we saw in Section 10, $T_{8,9}(-1)$ is imprimitive with $\mathbf{Q}[x]/T_{8,9}(-1, x)$ containing a degree twelve S_{12} subfield. It is different from the 106 degree twelve fields found in [5]. According to Section 10, $U_{8,9}(1, x)$ is imprimitive with $T_{8,9}(1, x)$ defining the corresponding subfield. This is indeed the case, with $T_{8,9}(1, x)$ having Galois group S_{28}. The absolute field discriminant is $2^{83}3^{54}$ which is smaller than the absolute field discriminants of the twenty-three degree 28 fields found in [5], the lowest discriminant there being $2^{92}3^{55}$.

The polynomial $T_{8,9}(2,x)$ factors as $(x-2)f_{35}(x)$ with $f_{35}(x)$ having Galois group all of S_{35}. The remaining nineteen points from (11.1) yield four A_{36} fields and fifteen S_{36} fields, all distinct. The polynomial $U_{8,9}(s,x)$ at the twenty-one s in (11.1) yields four A_{64} fields and seventeen S_{64} fields. Thus in summary, Galois groups behave completely generically, given the general expectations presented in Section 10.

A field with $S=\{3,5\}$. There are fewer possibilities for wild ramification at p in algebras of a given degree as p increases. For this reason $N(\{p_1,\dots,p_k\})$ decays as one p_i increases and the others are fixed. This explains why the threshold for exceptionalness $N(\{3,5\})=38$ is markedly less than the threshold $N(\{2,3\})=62$. Generally speaking, it is indeed harder to construct 3-5-number fields than 2-3 number fields by the method of three point covers, because the analog of (11.1) for any set of odd primes is empty.

Our source of an exceptional field with discriminant $\pm 3^a5^b$ is the cover $T_{25,27}(s,x)$ at the specialization point $s=1$. The polynomial $T_{25,27}(1,x)$ has degree 300 and Section 10 says the corresponding field has a degree 100 subfield. We have confirmed that indeed it does have a degree 100 subfield, the defining polynomial with roots β_{abc} being

$$T^{\rm red}_{25,27}(1,x)=x^{100}-625x^{99}+193,050x^{98}-39,288,375x^{97}+\cdots \tag{11.2}$$

The coefficients of x^j increase monotonically in size until the coefficient of x^{15} which has 83 digits; then they decrease monotonically with the constant term having 77 digits. Direct computation shows that its discriminant has the form

$$\mathrm{disc}_x(T^{\rm red}_{25,27}(1,x)=3^{614}5^{500}(23\cdot 137\cdot 25471\cdot 31482349\cdot C)^2. \tag{11.3}$$

Here $C\approx 4.2\times 10^{1006}$ is a non-prime having no prime factor $<10^{18}$. Here Jordan's criterion applies because $T^{\rm red}_{25,27}(1,x)$ is irreducible but modulo 2 factors as $71+14+12+3$; thus the Galois group is all of A_{100}.

12. Large degree exceptional fields with discriminant $\pm 2^a5^b$

In this final section, the focus is on five polynomials all with field discriminant of the form $\pm 2^a5^b$. Their degrees are $N=$ 7825, 7998, 8000, 8000, and 15875. The second one is expected by Section 10 to be 3-imprimitive and our calculations support this expectation. The genericity principle points to the remaining four having Galois group all of S_N, and we prove via Frobenius elements that this is indeed the case.

Manning's criterion for genericity. We continue with the set up of the previous section, except now we need to apply our considerations to polynomials in the range 7825 to 15875. At present, *Mathematica 6* does not factor polynomials $f(x)\in\mathbf{Z}[x]$ in this range. For very small primes p, it factors $f(x)$ in $\mathbf{F}_p[x]$ in approximately one-half hour for N near 7825 and two hours for N near 15875. To confirm genericity of polynomials with degrees of this order of magnitude, we need to use a stronger result that Jordan's simple criterion.

Results of Manning from around 1920 suffice for our purposes. One statement, a weakened and simplified version of Theorem 13.10 of the text [8], goes as follows. *Let G be a primitive subgroup of S_N. Suppose G contains an element of cycle type P^q1^k for p a prime. If $P\ge 2q-1$ and $k\ge 4q-1$ then G is A_N or S_N.*

To use Manning's criterion to confirm a Galois group is A_N or S_N, one typically uses the available primes $p_1 < p_2 < p_3 < \cdots$ to compute as many partitions λ_{p_i} as necessary. One uses these partitions in a three step process.

- First, one verifies transitivity, a necessary condition for primitivity. To do this, one looks at the set S_{p_i} of all partial sums of λ_{p_i}, taking j large enough so that $S_{p_1} \cap \cdots \cap S_{p_j} = \{0, N\}$.
- Second, one verifies that for every divisor $d \neq 1, N$ of N, the Galois group G is not in the wreath product of $S_{N/d}$ wr S_d. Cycles types appearing for $S_{N/d}$ wr S_d are exactly those whose parts are products $d_i \ell_{i,j}$, with the d_i running over the parts of a partition of d, and the $\ell_{i,j}$ running over a partition λ_i of N/d.
- Third, one finds a class in G which verifies Manning's large prime condition. Let $\lambda = \lambda_{p_i}$ be one of the available partitions. Let M be the least common multiple of the parts of λ. For all primes P exactly dividing M, let $\lambda^{M/P}$ be the indicated power of λ, so that a part ℓ of λ contributes $P^{\ell/P}$ to $\lambda^{M/P}$ if ℓ/P is an integer, and 1^ℓ to $\lambda^{M/P}$ otherwise. Then $\lambda^{M/P} = P^q 1^k$ is a candidate for satisfying the large prime condition.

In our five cases, the first four available primes are 3, 7, 11, 13. Always these four primes suffice for the first step. Always, any one of them suffices for the second step. Always, any one of them suffices for the third step.

Five large degree fields with $S = \{2, 5\}$. The threshold for exceptionalness in our final explicit case is $N(\{2, 5\}) = 49$. To construct exceptional fields, we use the covers $T_{125,128}(s, x)$ and $U_{125,128}(s, x)$. From our discriminant formulas, we know the corresponding discriminants have the form $\pm 2^* 5^* s^* (s-1)^* d_3^F(s)$ with $d_3^T(s) = s + 1$ and $d_3^U(s) = s + 27$.

Special points which give algebras with discriminants of the form $\pm 2^a 5^b$ are $s = 1$, -1, $4/5$, and $5/4$ for $T_{125,128}(s, x)$ and $s = 5$ for $U_{125,128}(s, x)$. The degrees of these algebras are 7875, 7998, 8000, 8000, and 15875. The points $s = 5/4$, $s = 4/5$ introduce factors of 3^2 into the polynomial discriminant but these factors necessarily drop out in the field discriminant because the 3-adic proximity $\mathrm{ord}_3(s + 1) = 2$ is a multiple of the ramification index $e = 2$.

As a convenient simpler parallel case, we can replace $(125, 128)$ by $(5, 8)$ and use the same specialization points. Then from $T_{5,8}(s, x)$ at $s = 1$, -1, $4/5$, and $5/4$ we get fields of degree 15, 18, 20, 20. The second has a degree six subfield with Galois group S_6 and the remaining three have Galois groups S_{15}, S_{20}, S_{20}, all as expected. The field discriminants are respectively $2^{36}5^{15}$, $2^{43}5^{17}$, $2^{59}5^{36}$, and $2^{59}5^{37}$. Similarly for $U_{5,8}(s, x)$ at $s = 5$ we get a field of degree 35, field discriminant $-2^{89}5^{67}$, and Galois group all of S_{35}.

Table 12.1 gives the factorization partition of these five polynomials for the primes 3, 7, 11, and 13. In all cases except for the second-printed case $s = -1$, this information is enough to conclude genericity by Manning's criterion. The upper left corner of each subtable gives the discriminant class of the two corresponding polynomials, calculated using Theorem 4.1 for $s \neq \pm 1$ and other results of Section 4 for $s = \pm 1$. In each case this discriminant class is different from 1, proving that the Galois group is symmetric rather than alternating.

One has $T_{m,n}(-1, x) = T_{m,n}(-1, x)^*(x + 1)^2$ for both $(m, n) = (5, 8)$ and $(125, 128)$. In the second frame of Table 12.1, parts are grouped either in ones,

5	$T_{5,8}(1,x)$	$T_{125,128}(1,x)$	degree = 7875
3−	9 6	4657 1976 475 469 153 68 43 22 9 3	
7−	10 2 2 1	5635 2224 7 6 2 1	
11+	15	6453 2224 7 6 2 1	
13−	12 3	5211 1580 658 259 128 25 12 2	

10	$T_{5,8}(-1,x)^*$	$T_{125,128}(-1,x)^*$	degree = 7998 = 3 · 2666
3+	(10 5) (2 1)	2131^3 1095 (276 138) 90 3 (2 1)	
7−	18	3480 (1276 638) (800 400) 317^3 (282 141) (16 8) 2^3	
11−	6^3	(5038 2519) (146 73) 74^3	
13+	(10 5) (2 1)	6744 486 (440 220) 75 (8 4) 6 3^3 2^3	

5	$T_{5,8}(4/5,x)$	$T_{125,128}(4/5,x)$	degree = 8000
3−	10 5 2 2 1	2131 2131 2131 1095 276 138 90 3 2 2 1	
7+	15 2 2 1	3772 2046 923 814 294 58 52 17 14 6 2 2	
11−	10 9 1	4758 2048 537 343 157 62 44 39 7 4 1	
13−	10 7 1 1 1	5009 1550 1164 157 87 19 12 1 1	

10	$T_{5,8}(5/4,x)$	$T_{125,128}(5/4,x)$	degree = 8000
3+	10 5 2 1 1 1	2131 2131 2131 1095 276 138 90 3 2 1 1 1	
7−	9 6 3 1 1	5978 1284 700 33 2 2 1	
11−	20	7020 856 85 36 3	
13+	15 5	4740 1715 639 527 204 171 2 2	

−10	$U_{5,8}(5,x)$	$U_{125,128}(5,x)$	degree = 15875
3−	27 8	10194 3365 2123 155 20 10 5 3	
7+	35	7332 2492 1642 1388 1077 1011 818 72 24 10 9	
11+	10 10 6 6 3	9784 3238 1272 648 480 143 139 133 17 12 9	
13+	15 12 5 2 1	6808 4493 3803 626 74 39 13 8 6 3 2	

TABLE 12.1. Factor partitions for five large-degree polynomials from the case $(m,n) = (125,128)$ at the small primes $p = 3$, 7, 11, and 13. A parallel case with $(m,n) = (5,8)$ is also treated to the left.

twos, or threes, and expressed as $3a$, $(2a,a)$, or a^3, showing consistency with the expectation of 3-imprimitivity. Each of these forms yields a in the corresponding partition of 6 or 2666, proving that $\mathbf{Q}[x]/T_{m,n}(-1,x)^*$ has a subfield of index three, then its associated Galois group is indeed S_{2666} by Manning's criterion.

As a side point, for both $(m,n) = (5,8)$ and $(125,128)$, one has that the reduction of $T_{m,n}(s,x)$ to $\mathbf{F}_3[x]$ is independent of $s \in \{-1, 5/4, 4/5\}$ and contains the singular factor $(x+1)^2$. This accounts for the near-agreement of the three corresponding 3-adic factor partitions.

Returning to our largest example, our degree 15875 polynomial is

$$U_{125,128}(5,x) = (x-2)^3 u_{62.5}(x)^{256} - 5(x+2)^{125} u_{64}(x)^{250}.$$

Its existence depends not only on the general theory of Chebyshev covers but also on the two ABC-triples $5^3 + 3 = 2^7$ and $3^3 + 5 = 2^5$. Remarkably, despite the presence of the prime 3 in both these triples, not even the polynomial discriminant of $U_{125,128}(1,x)$ is divisible by 3.

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