# What is a motive?

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# Section 1. Preliminary general remarks
# A historical parallel: Galois and Grothendieck

Around 1830, Galois used **finite groups** to study solutions of *univariate polynomial equations* $f(x) = 0$.

In the 1960s, Grothendieck used **reductive algebraic groups** to study solutions of *general polynomial equations* $f(x_1, \ldots, x_n) = 0$.

In both cases, the work was not published in a timely way, and was fully appreciated only much later.

# Miscellaneous notes

**1.** We are using a modification of Grothendieck's original definition due to André. This change makes the basic definitions *unconditional*.

**2.** In particular, we are always talking about pure motives rather than mixed motives. We are not considering modern enhancements involving Chow groups, $K$-theory, derived categories, and so on.

**3.** A central player in the full classical theory is the category $\mathcal{M}(K, E)$ of motives "over $K$ with coefficients in $E$" with $K$ and $E$ subfields of $\mathbb{C}$. We will simplify throughout by taking $K = \mathbb{Q}$.

# 0- and 1-dimensional varieties

Many researchers are experts in 0- and/or 1-dimensional varieties. This expertise is a tremendous asset in trying to learn the theory of motives. Examples:

- For 0-dimensional varieties, the theory of motives reduces to the theory of continuous linear representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $GL_n(\mathbb{C})$. The usual decomposition groups $D_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are very close to being the decomposition groups $WD_p$ needed for the full motivic theory.
- For 1-dimensional varieties, the theory of motives is very close to the theory of Jacobians. Objects such as the endomorphism algebra of a Jacobian or the $\ell$-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ coming from $\ell$-primary torsion in $J(\overline{\mathbb{Q}})$ have direct generalizations in the full motivic theory.

Looked at motivically, the expected situation for general varieties is not too different from the more established special case of varieties of dimension $\leq 1$.

# Section 2. Cohomology and cycles

Consider smooth projective varieties $X$ over $\mathbb{Q}$. For each such $X$, one has the associated compact manifold $X(\mathbb{C})$. Consider the usual cohomology spaces

$$H^*(X(\mathbb{C}), \mathbb{Q}) = \sum_{w=0}^{2\dim(X)} H^w(X(\mathbb{C}), \mathbb{Q}).$$

For $w = 2j$ even, one has the subspace spanned by the fundamental classes of codimension $j$ subvarieties defined over $\mathbb{Q}$:

$$H^w(X(\mathbb{C}), \mathbb{Q})^{\mathrm{alg}} \subseteq H^w(X(\mathbb{C}), \mathbb{Q})$$

The interplay of all cohomology and the (typically very small) part represented by algebraic cycles is central to the theory of motives.

# $H^w(X(\mathbb{C}), \mathbb{Q})^{\mathrm{alg}} \subseteq H^w(X(\mathbb{C}), \mathbb{Q})$ for $\dim(X) = 0$

Let $f(x) \in \mathbb{Q}[x]$ be a separable degree $n$ polynomial with root set $X(\mathbb{C}) \subset \mathbb{C}$. Let $f(x) = f_1(x) \cdots f_d(x)$ be its factorization into irreducibles, with $f_j(x)$ having root set $X_j(\mathbb{C})$. Then

$$X(\mathbb{C}) = X_1(\mathbb{C}) \coprod \cdots \coprod X_d(\mathbb{C}).$$

Very simply,

- $H^0(X(\mathbb{C}), \mathbb{Q})$ is the space of $\mathbb{Q}$-valued functions on $X(\mathbb{C})$.
- $H^0(X(\mathbb{C}), \mathbb{Q})^{\mathrm{alg}}$ is the subspace of functions constant on each $X_j(\mathbb{C})$.

Note that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts naturally on $X(\mathbb{C}) = X(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$. Its orbits are exactly the $X_j(\mathbb{C})$. Its invariants in $H^0(X(\mathbb{C}), \mathbb{Q})$ form exactly the subspace $H^0(X(\mathbb{C}), \mathbb{Q})^{\mathrm{alg}}$.

# $H^w(X(\mathbb{C}),\mathbb{Q})^{\mathrm{alg}} \subseteq H^w(X(\mathbb{C}),\mathbb{Q})$ for $\dim(X) \geq 1$

Let $X$ be a curve with $X(\mathbb{C})$ connected of genus $g$. Then

| $w$ | $\dim_{\mathbb{Q}}(H^w(X(\mathbb{C}),\mathbb{Q})^{\mathrm{alg}})$ | $\dim_{\mathbb{Q}}(H^w(X(\mathbb{C}),\mathbb{Q}))$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | $2g$ |
| 2 | 1 | 1 |

For $\dim(X) > 1$ there are not known to be enough algebraic cycles to support the Grothendieck formalism. André's modification is to introduce an intermediate space of *quasialgebraic* cycles:

$$H^w(X(\mathbb{C}),\mathbb{Q})^{\mathrm{alg}} \overset{*}{\subseteq} H^w(X(\mathbb{C}),\mathbb{Q})^{\mathrm{qalg}} \subseteq H^w(X(\mathbb{C}),\mathbb{Q}).$$

Working with quasialgebraic cycles, the Grothendieck definitions go through unconditionally. It is expected, although not needed for the basic theory, that equality always holds at (*).

# Cycles on self-powers $X^k$

The Künneth formula says that cohomology behaves simply with respect to products in general and self-powers in particular:

$$H^*(X^k(\mathbb{C}),\mathbb{Q}) = H^*(X(\mathbb{C}),\mathbb{Q})^{\otimes k}.$$

However "new" quasialgebraic cycles very typically appear on self-products:

$$H^*(X^k(\mathbb{C}),\mathbb{Q})^{\mathrm{qalg}} \supseteq \left(H^*(X(\mathbb{C}),\mathbb{Q})^{\mathrm{qalg}}\right)^{\otimes k}.$$

Sometimes these new cycles have a tautological nature, e.g. the diagonal $\Delta \subset X^2$. Sometimes these new cycles are very specific to the variety $X$ being studied, e.g. the graph $\Gamma_f \subset X^2$ of a map $f : X \to X$.

# Section 3. Motives and motivic Galois groups

## Definition

Let $X$ be a smooth projective variety over $\mathbb{Q}$. Its *special motivic Galois group* $G_X^1$ is the group of automorphisms of the vector space $H^*(X(\mathbb{C}), \mathbb{Q})$ which fix all the spaces $H^*(X^k(\mathbb{C}), \mathbb{Q})^{\mathrm{qalg}}$.

By definition, $G_X^1$ is an algebraic group over $\mathbb{Q}$, and one has

$$H^*(X^k(\mathbb{C}), \mathbb{Q})^{G_X^1} \supseteq H^*(X^k(\mathbb{C}), \mathbb{Q})^{\mathrm{qalg}}. \tag{3.1}$$

## Theorem

*Equality holds in* (3.1) *for all* $k$.

## Corollary

$G_X^1$ *is reductive.*

# $G_X^1$ for $\dim(X) \leq 1$

**Dimension 0:** For $X = \mathrm{Spec}\ \mathbb{Q}[x]/f(x)$ as before, the spaces $H^0(X^k(\mathbb{C}), \mathbb{Q})^{\mathrm{alg}}$ are easily computed by factoring resolvents of $f(x)$. From the presence of many algebraic cycles, one gets

$$G_X^1 \subseteq S_n.$$

In fact, $G_X^1$ is exactly the ordinary Galois group of $f(x)$.

**Dimension 1:** For $X$ a curve as before, the diagonal $\Delta \subset X^2$ gives rise to an alternating pairing on $H^1(X(\mathbb{C}), \mathbb{Q})$. This pairing leads to

$$G_X^1 \subseteq Sp_{2g},$$

with generic equality. In the non-generic case, extra cycles for $k = 2$ come from endomorphisms of the Jacobian. Extra cycles for $k = 4$, 6, 8, ... come mainly from potential endomorphisms of the Jacobian, but also can come from more exotic sources.

# Two mostly formal steps

**Tate twists.** So far we have been trivializing Tate twists. Repeating the definitions (not done here!) without trivializing Tate twists gives the full motivic Galois group $G_X$. There is no change for $\dim(X) = 0$. An extra $\mathbb{G}_m$ is tacked on for $\dim(X) \geq 1$; e.g., for a generic elliptic curve $X$, $G_X^1 = SL_2$ and $G_X = GL_2$.

**Projective limits.** Taking a projective limit over all $X$ gives a pro-reductive group $\mathbb{G}$ over $\mathbb{Q}$. It fits in an exact sequence

$$\mathbb{G}^0 \hookrightarrow \mathbb{G} \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

The kernel $\mathbb{G}^0$ is conjecturally connected. Thus Grothendieck/André's motivic Galois theory is literally an extension of Galois' classical Galois theory; however the new part $\mathbb{G}^0$ is quite different.

# Motives

Fix a field $E \subseteq \mathbb{C}$.

## Definition

*The category of motives $\mathcal{M}(\mathbb{Q}, E)$ is the category of representations of $\mathbb{G}$ on finite-dimensional $E$-vector spaces. The motivic Galois group of a motive $M \in \mathcal{M}(\mathbb{Q}, E)$ is the image $G_M$ of $\mathbb{G}$ on $M$.*

One has a gradation by weight: $\mathcal{M}(\mathbb{Q}, E) = \bigoplus_w \mathcal{M}(\mathbb{Q}, E)^w$. Concretely,

1. Any $H^w(X(\mathbb{C}), E)$ is a weight $w$-motive.
2. Any $\mathbb{G}$-stable subspace of $H^w(X(\mathbb{C}), E)$ is a weight $w$ motive.
3. Any irreducible motive is a Tate twist of a motive as in (2).

The group $G_M$ is not just a formal object. Rather it coordinates the arithmetic of $M$. Conjecturally, this coordination of *a priori* disparate invariants is extremely tight.

# Section 4. Role in the Langlands program

In extreme brevity:

1. Modulo some technical conjectures, an irreducible rank $n$ motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{C})$ gives rise to an $L$-function
$$L(M, s) = \sum \frac{a_n}{n^s} = \prod_p \frac{1}{f_p(p^{-s})},$$
with a completion $\Lambda(M, s) = N^{s/2} L_\infty(M, s) L(M, s)$.
2. $\Lambda(M, s)$ should be either $\zeta(s - w/2)$ or an entire function. It should satisfy a functional equation $\Lambda(M, s) = \epsilon \Lambda(\overline{M}, w + 1 - s)$.
3. These motivic $L$-functions together should coincide with the set of automorphic $L$-functions $\Lambda(\rho, s)$ for cuspidal algebraic automorphic representations $\rho$ of $GL_n(\text{Adeles}_{\mathbb{Q}})$.

The connection with automorphic representations makes it reasonable to explicitly classify objects in $\mathcal{M}(\mathbb{Q}, \mathbb{C})$. E.g., irreducible $M \in \mathcal{M}(\mathbb{Q}, \mathbb{C})$ of rank one and two should respectively correspond to Tate twists of Dirichlet motives $M_\chi$ and modular motives $M_f$.