Complete hypergeometric L-functions

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reporting on ongoing project with

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A standard setting

In general, suppose given a smooth projective variety X over $\mathbb{Q}$ and a motive $M \subseteq H^w(X(\mathbb{C}), \mathbb{Q})$. One wants to know:

1. The Hodge vector $(h^{w,0}, h^{w-1,1}, \ldots, h^{1,w-1}, h^{0,w})$.

2. The Frobenius polynomials $F_p(x)$ and hence the local L-factors $L_p(M,s) = F_p(p^{-s})^{-1}$ for all primes p outside a finite bad set S.

3. The remaining $L_p(M,s) = F_p(p^{-s})^{-1}$ and the conductor $N = \prod_p p^{c_p}$.

4. Analytic properties of $\Lambda(M,s) = N^{s/2} L_\infty(M,s) \prod_p L_p(M,s)$ including the functional equation $\Lambda(M,s) = \pm\Lambda(M, w+1-s)$.

The talk will be about hypergeometric motives $M = H(A,B,t)$ and the current state of 1-4 for them.

Hypergeometric families: defining data

Families of hypergeometric motives are indexed by ordered pairs $(f(x), g(x))$ of relatively prime monic polynomials in $\mathbb{Z}[x]$ sharing a common degree d and vanishing only on roots of unity.

Given such a pair, we write

$$f(x) = \prod_i \Phi_{a_i}(x),$$

$$g(x) = \prod_i \Phi_{b_i}(x).$$

We notationally emphasize $A = [a_1, \dots]$ and $B = [b_1, \dots]$.

Example with $d = 5$:

$$f(x) = \Phi_2(x)\Phi_4(x)^2 = (x+1)\left(x^2+1\right)^2,$$

$$g(x) = \Phi_1(x)\Phi_3(x)^2 = (x-1)\left(x^2+x+1\right)^2.$$

Here $A = [2, 4, 4]$ and $B = [1, 3, 3]$.

Hypergeometric families: monodromy matrices

Given $(f(x), g(x))$ as before, let m_f and m_g be their companion matrices. Define

$$(m_0,\ m_1,\ m_\infty) = (m_f,\ m_f^{-1}m_g,\ m_g^{-1}).$$

By construction, $m_0 m_1 m_\infty = 1$.

Example, continued: m_0, m_1, and m_∞ are

$$\begin{pmatrix} & & & & -1 \\ 1 & & & & -1 \\ & 1 & & & -2 \\ & & 1 & & -2 \\ & & & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & -1 \\ & & 1 & & -3 \\ & & & 1 & -2 \\ & & & & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & 1 & & & \\ -1 & & 1 & & \\ 1 & & & 1 & \\ 1 & & & & 1 \\ 1 & & & & \end{pmatrix}$$

In general, m_0 and m_∞ are regular, but $m_1 - 1$ has rank one.

Hypergeometric motives: “definition”

Let $T = \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ with base point $\star = 1/2$.

The map $\pi_1(T, \star) \to GL_d(\mathbb{Q})$ sending the standard generator γ_τ to the matrix m_τ gives a local system on T, i.e., a family of d-dimensional rational vector spaces $H(A, B, t)$ indexed by $t \in T$. In our example, $H(A, B, t) = H([2, 4, 4], [1, 3, 3], t)$ is a local system of 5-dimensional $\mathbb{Q}$-vector spaces.

There is a natural family of smooth complex projective varieties $X(A, B, t)$ such that $H(A, B, t)$ appears in some $H^k(X(A, B, t), \mathbb{Q})$ [BCM]. The family is defined over $\mathbb{Q}$, giving motives $H(A, B, t)$ for $t \in \mathbb{Q}^\times - \{1\}$.

I will skip defining $X(A, B, t)$. The philosophy is that final answers to 1-4 should be *expressible* in terms of A, B, t alone. They should be in substantial measure *understandable* in terms of (m_0, m_1, m_∞).

1. Hodge vector: combinatorial computation

The Hodge vector of the family $H(A, B)$ can be computed by the following combinatorial procedure.

1. Draw the roots $\exp(2\pi i \alpha_j)$ of $f(x)$ and $\exp(2\pi i \beta_j)$ of $g(x)$ on the unit circle.

2. Draw a zig-zag function over the unit circle flattened to $[0, 1)$, going up one when you encounter an α_j and down one when you encounter a β_j.

3. The Hodge vector $(h^{w,0}, \ldots, h^{0,w})$, normalized by $h^{w,0} = h^{0,w} \neq 0$, is such that $h^{p,w-p}$ counts the up steps at height p.

Our family $H([2, 4, 4], [1, 3, 3])$ has Hodge vector $(1, 3, 1)$.

1. Hodge vector: possibilities

There are $2^{\lfloor d/2 \rfloor}$ *a priori* possibilities for $\vec{h}$ in degree d.

- One extreme: complete intertwining yields $\vec{h} = (d)$.
- An intermediate case yields $\vec{h} = (1, d-2, 1)$.
- The other extreme: complete separation yields $\vec{h} = (1, 1, \ldots, 1, 1)$.

A given Hodge vector can occur for many families:

d :	1	2	3	4	5	6	7	8	9	$\cdots$
(d)	1	2	2	7	4	13	11	31	7	$\cdots$
$(1, d-2, 1)$			6	15	31	56	53	120	95	$\cdots$
$(1, \ldots, 1)$	1	6	6	25	25	73	73	184	184	$\cdots$

In degrees $d \leq 20$, the only possibility which does not actually occur is $\vec{h} = (6, 1, 1, 1, 2, 1, 1, 1, 6)$.

2. Good factors and 3a tame factors

Fix $H(A, B, t)$. Suppose a prime p

1. does not divide a member of A or B
2. does not divide $\mathrm{num}(t)$, $\mathrm{num}(t-1)$, or $\mathrm{denom}(t)$.

Then $H(A, B, t)$ has good reduction at p and the degree d Frobenius polynomials $F_p(x)$ can be calculated via a remarkable trace formula involving Gauss sums [Greene, Katz, BCM].

If p satisfies 1 but not 2, then $H(A, B, t)$ is tamely ramified at p. Again $F_p(x)$ can be calculated. The contribution to the conductor is p^{c_p} where $c_p = d - \mathrm{degree}(F_p(x))$.

In both cases, the local L-factor is

$$L_p(H(A, B, t), s) = \frac{1}{F_p(p^{-s})}.$$

2. Good and tame L-factors: Example

As a sample of *Magma*'s functionality (with answer prettified):

```
>H := HypergeometricData([2,4,4],[1,3,3]);
>[EulerFactor(H,10/3,p):  p in PrimesInInterval(5,17)];
```

5	$1-5x$
7	$(1-7x+7^2x^2)(1+2x+7^2x^2)$
11	$(1-11x)(1-6x+15\cdot 11x^2-6\cdot 11^2x^2+11^4x^4)$
13	$(1-13x)^2(1+13x)(1+13x+13^2x^2)$
17	$(1-17x)(1-8x+14\cdot 17x^2-8\cdot 17^2x^3+17^4x^4)$

The good factors are all of the form

$$F_p(x) = (1-\left(\tfrac{-21}{p}\right)px)(1+a_px+b_ppx^2+a_pp^2x^3+p^4x^4)$$

This form reflects the motivic Galois group $G = GO_5$, the determinant $\det(M)$, and the Hodge vector $(1,3,1)$.

3b. Wild L-factors via congruences and Belyi maps

For general motives M, contributions p^{c_p} to the conductor from wild primes p can be expected to be complicated.

Similarly, wild L-factors $L_p(M,s) = F_p(p^{-s})^{-1}$ are complicated (although identically 1 in the totally ramified case).

A key fact is that *much of this information can be read off from mod* ℓ *representations for any* $\ell \neq p$.

We have Belyi maps $Y \to \mathbb{P}^1$ giving mod 2 representations for most cases in degree ≤ 7, and mod 3 representation for most cases in degree ≤ 5.

These covers allow determination of wild factors, and thus complete L-functions $\Lambda(H(A,B,t),s)$, in many cases in low degree. It's on our agenda to incorporate this fully into *Magma*.

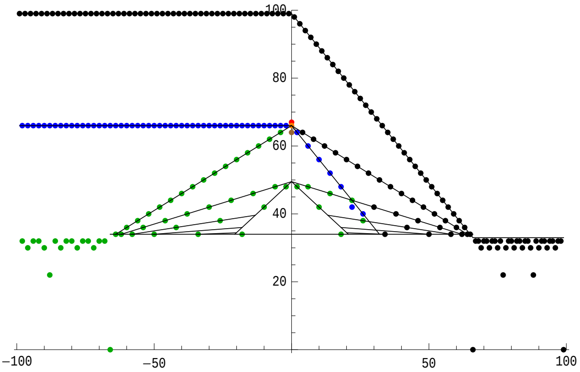
3b. Wild L-factors: a large degree example

In favorable cases, wild ramification can be completely analyzed in large degree as well. For example, consider the large degree family $H([\overbrace{2,\ldots,2}^{33}],[\overbrace{1,\ldots,1}^{33}],t)$. It has the most complicated Hodge vector $(\overbrace{1,\ldots,1}^{33})$. To obtain complete L-functions, we are only missing information at 2. Fortunately, there is a chain of congruences:

$$\begin{aligned} H([2^{33}],[1^{33}],t) &\overset{11}{\equiv} H([22,22,22,2,2,2],[11,11,11,1,1,1],t) \\ &\overset{3}{\equiv} H([66,22,6,2],[33,11,3,1],t) \quad (\vec{h}=(33)). \end{aligned}$$

This chain says that c_2 for $H([2^{33}],[1^{33}],t)$ is the same for c_2 of the Artin motive $H([66,22,6,2],[33,11,3,1],t)$ whenever the latter is totally wild.

The conductor exponent c_2 for $H([66, 22, 6, 2], [33, 11, 3, 1], t)$ for $t = u2^k$ is indicated by the picture:



For most k, it is independent of u (black). For some k it depends on whether $u \equiv 3$ (4) (blue) or $u \equiv 1$ (4) (green).

For all $H(A, B, t)$ and all wild p, the picture seems to be qualitatively similar.

4. L-functions: numerical certification

We informally say that the completion $\Lambda(M,s)$ of a correct partial L-function $L_S(M,s)$ has been *numerically certified* if it passes *Magma*'s `CheckFunctionalEquation` to high precision.

Our expectation is that a numerically certified $\Lambda(M,s)$ indeed has all its extra factors correct and indeed satisfies the expected analytic continuation and functional equation.

From hypergeometric motives we get many numerically certified $\Lambda(M,s)$ with a broad range of Hodge vectors $(h^{w,0},\ldots,h^{0,w})$ and full motivic Galois group GSp_d or GO_d.

4. L-functions: an example

The specialization point $t = 1$ gives particularly interesting motives where formulas are slightly different. For example:

The motive $M = H([2^{16}], [1^{16}], 1)$ has Hodge vector $(1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1)$, a certified-to-10-digits $\Lambda(M, s)$, with conductor 2^{15}, sign 1, order of central vanishing 2, and $L''(M, 8) \approx 7.851654518$.

The first two Frobenius polynomials (two seconds and thirty seconds):

$$\begin{aligned}
F_3(x) \quad = \quad & (1 - 268 \cdot 3x + 204193 \cdot 3^4 x^2 - 1001800 \cdot 3^9 x^3 + 204193 \cdot 3^{19} x^4 - 268 \cdot 3^{31} x^5 + 3^{45} x^6) \\
& (1 + 2992 \cdot x + 39116 \cdot 3^4 x^2 - 7596496 \cdot 3^6 x^3 - 203836426 \cdot 3^{12} x^4 \\
& \qquad -7596496 \cdot 3^{21} x^5 + 39116 \cdot 3^{34} x^6 + 2992 \cdot 3^{45} x^7 + 3^{60} x^8)
\end{aligned}$$

$$\begin{aligned}
F_5(x) \quad = \quad & (1 + 1614 \cdot 5^3 x + 28284579 \cdot 5^4 x^2 + 1394686516 \cdot 5^9 x^3 + 28284579 \cdot 5^{19} x^4 + 1614 \cdot 5^{33} x^5 + 5^{45} x^6) \\
& (1 - 41208 \cdot x - 44999364 \cdot 5^3 x^2 - 22376708712 \cdot 5^6 x^3 + 3926679014806 \cdot 5^{12} x^4 \\
& \qquad -22376708712 \cdot 5^{21} x^5 - 44999364 \cdot 5^{33} x^6 - 41208 \cdot 5^{45} x^7 + 5^{60} x^8)
\end{aligned}$$

4. L-functions: the example, continued

The splitting $M = M_6 \oplus M_8$ is known *a priori* from a joint symmetry $t \leftrightarrow 1/t$ and $2 \leftrightarrow 1$. The Hodge vectors of the summands are

$$(0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0),$$
$$(1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1).$$

The two Frobenius polynomials suffice to prove that the motivic Galois group of the two factors are GSp_6 and GSp_8.

Q1. Since $L_2(M, s) = 1$, there are only two possibilities for $(\mathrm{cond}(M_6), \mathrm{cond}(M_8))$, namely $(2^6, 2^9)$ or $(2^7, 2^8)$. Which is it?

Q2. There are only three possibilities for $(\mathrm{rank}(M_6), \mathrm{rank}(M_8))$, namely $(2, 0)$, $(1, 1)$, or $(0, 2)$. Which one is correct?

Some references

Hypergeometric Motives, with Fernando Rodriguez Villegas and Mark Watkins, in preparation. Several presentations by each of us available online.

Finite hypergeometric functions, by Frits Beukers, Henri Cohen, and Anton Mellit. ArXiv May 12, 2015.

Hypergeometric functions over finite fields, by John Greene, Trans. Amer. Math. Soc. **301** (1987), 77-101.

Exponential Sums and Differential Equations, by Nicholas M. Katz, Annals of Math Studies, **124**, (1990) is an early work emphasizing motivic aspects of hypergeometric functions.

Some references, continued

Variations of Hodge Structure for Hypergeometric Differential operators and parabolic Higgs bundles, by Roman Fedorov, ArXiv May 7, 2015, has the Hodge number formula. Antecedents include works of Terasoma, Corti, Golyshev, Dettweiler, and Sabbah.

Monodromy for the hypergeometric function ${}_nF_{n-1}$, by Frits Beukers and Gert Heckman, Invent. Math. **95** (1989), 325-354, definitively treats the case of complete intertwining.

The HGM package in *Magma* is by Mark Watkins. The L-function package is by Tim Dokchitser.