An LMFDB perspective on motives

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Classical language is adequate for studying L-functions associated to 0- and 1-dimensional varieties.

Q. What is a good language for studying L-functions associated to general varieties?

A. The language of motives.

Motives were first defined under a still-unproven "standard conjecture" by Grothendieck in the mid 1960s. Fundamental results inspired by motives were obtained in the next decades, especially by Deligne, but motives themselves were regarded by many as illicit. The unconditional definition we use is due to André in 1994. There have been many advances in strikingly diverse directions in recent decades.

Partial L-functions $L_S((X,w),s)$. Let X be a smooth projective variety over $\mathbb{Q}$ of dimension d. Let $b_w = \dim_{\mathbb{Q}} H^w(X(\mathbb{C}),\mathbb{Q})$.

For p a prime of good reduction,

$$|X(\mathbb{F}_{p^k})| = \sum_{w=0}^{2d} \sum_{j=1}^{b_w} (-1)^w \alpha_{w,j}^k$$

with $|\alpha_{w,j}| = p^{w/2}$ (Weil, Dwork, Grothendieck, Deligne).

Define

$$f_{X,w,p} = \prod_{j=1}^{b_w} (1 - \alpha_{w,j} x) \in \mathbb{Z}[x],$$

$$L_p((X,w),s) = \frac{1}{f_{X,w,p}(p^{-s})}.$$

For S a finite set of primes including all bad primes, define

$$L_S((X,w),s) = \prod_{p \notin S} L_p((X,w),s).$$

We want to understand the $L_S((X,w),s)$.

Desiderata. Inspired by extensive experience for $\dim(X) \leq 1$, we'd like a natural factorization into irreducibles

$$L_S((X,w),s) = \prod_M L_S(M,s).$$

For each irreducible, we'd like

A. Local factors $L_p(M,s)$, with $p \in S$.

B. A conductor $N = \prod_{p \in S} p^{n_p}$.

C. A Gamma-factor $L_\infty(M,s)$.

D. Analytic properties of $L(M,s) = \prod_v L_v(M,s)$.

E. Equidistribution properties of coefficients.

F. Interpretations of special values $L(M,n)$.

Special Motivic Galois groups. One has

$$H^*(X(\mathbb{C}),\mathbb{Q})^{\otimes k} = H^*(X^k(\mathbb{C}),\mathbb{Q})$$

by the Künneth theorem. Inside $H^*(X^k(\mathbb{C}),\mathbb{Q})$ one has spaces of classes represented by algebraic and "quasialgebraic" cycles

$$H^*(X^k(\mathbb{C}),\mathbb{Q})^{\mathsf{alg}} \subseteq H^*(X^k(\mathbb{C}),\mathbb{Q})^{\mathsf{qalg}}. \quad (\star)$$

Definition. *The* special motivic Galois group G^1_X *is the group of automorphisms of the vector space* $H^*(X(\mathbb{C}),\mathbb{Q})$ *which fixes all the spaces* $H^*(X^k(\mathbb{C}),\mathbb{Q})^{\mathsf{qalg}}$*.*

Grothendieck's standard conjecture would give equality in ($\star$), but for the present quasialgebraic cycles are necessary for a good unconditional theory.

By definition, G^1_X is an algebraic group over $\mathbb{Q}$. Equality always holds in

$$H^*(X^k(\mathbb{C}),\mathbb{Q})^{G^1_X} \supseteq H^*(X^k(\mathbb{C}),\mathbb{Q})^{\mathsf{qalg}},$$

making G^1_X reductive.

The cases $\dim(X) \leq 1$**.**

Points. For X the spectrum of a degree n number field $K = \mathbb{Q}[x]/f(x)$, the inclusion

$$H^0(X^k(\mathbb{C}), \mathbb{Q})^{\text{alg}} \subseteq H^0(X^k(\mathbb{C}), \mathbb{Q})$$

is easily computed by factoring resolvents of $f(x)$. One gets

$$G_X^1 \subseteq S_n$$

with G_X^1 being exactly the ordinary Galois group of $f(x)$.

Curves. For X a geometrically connected curve of genus g, the pairing on $H^1(X(\mathbb{C}), \mathbb{Q})$ leads to

$$G_X^1 \subseteq Sp_{2g},$$

with generic equality. In the non-generic case, extra cycles for $k = 2$ come from endomorphisms of the Jacobian. Extra cycles for $k = 4$, 6, 8, ... come mainly from potential endomorphisms of the Jacobian, but also can come from more exotic sources.

Full motivic Galois groups. We have been trivializing Tate twists. Repeating the definitions without trivializing Tate twists gives the full motivic Galois group G_X. There is no change for $\dim(X) = 0$ and an extra G_m is tacked on for $\dim(X) \geq 1$. E.g., for a generic elliptic curve X, $G_X^1 = SL_2$ and $G_X = GL_2$.

Projective limits. Taking a projective limit over all X gives a pro-reductive group over $\mathbb{Q}$, coming with a surjection

$$\mathbb{G} \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

The kernel is conjecturally connected.

Representations. The category of motives $\mathcal{M}$ is the category of representations of $\mathbb{G}$ on finite-dimensional $\mathbb{Q}$-vector spaces. It is graded by weight:

$$\mathcal{M} = \bigoplus_{w \in \mathbb{Z}} \mathcal{M}^w.$$

Here $H^w(X(\mathbb{C}), \mathbb{Q}) \in \mathcal{M}^w$. Attention naturally focuses on irreducible motives M and their motivic Galois groups $G_M = \mathrm{Image}(\mathbb{G})$.

Connections with classical Galois theory.

I. From ℓ-adic representations on $H^*(X(\mathbb{C}), \mathbb{Q}_\ell)$ one has canonical sections

$$\mathbb{G}(\mathbb{Q}_\ell) \longleftarrow \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

The Tate conjecture says that the image is open in every reductive quotient.

II. The local groups

$$\{1\} \subset P_p \overset{\widehat{\mathbb{Z}}^p}{\subset} I_p \overset{\widehat{\mathbb{Z}}}{\subset} D_p \text{ inside } \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

have motivic analogs, some aspects of this theory being conjectural. Roughly, one has

$$\{1\} \subset P_p \overset{\widehat{\mathbb{Z}}^p \times G_a}{\subset} \tilde{I}_p \overset{\mathbb{Z}^{\mathsf{tor}}}{\subset} \tilde{D}_p \text{ inside } \mathbb{G}.$$

So the unramified quotient is now a toroidal rather than profinite completion of $\mathbb{Z}$. The tame subquotient includes a G_a to allow for multiplicative reduction. The wild subgroup P_p, together with its much-studied filtration, is exactly the same! *The slides assume the conjectures, the verbal comments will explain how things become more technical without them.*

III. The classical decomposition group $D_\infty = \{1, \sigma\}$ becomes the motivic $\tilde{D}_\infty = \mathbb{C}^\times . D_\infty$. The new part $\mathbb{C}^\times$ acts on $H^*(X(\mathbb{C}), \mathbb{C}) = \bigoplus H^{p,q}$ with z acting on $H^{p,q}$ as $z^p \overline{z}^q$. The Hodge conjecture says that $\mathbb{C}^\times$ is $\mathbb{Q}$-Zariski dense in the neutral component of $\mathbb{G}$.

Example. The hypergeometric motive

$$M = H([3^3], [2^6], 1)$$

is a summand of H^5 of a five-dimensional variety and is on the LMFDB. Local invariants:

$$(h^{5,0}, h^{4,1}, h^{3,2}, h^{2,3}, h^{1,4}, h^{0,5}) = (1, 1, 0, 0, 1, 1)$$

p	c_p	$f_{M,p}(x)$
2	6	1
3	5	1
5		$1 + 6x - 5 \cdot 249x^2 + 5^5 \cdot 6x^3 + 5^{10}x^4$
7		$1 + 7 \cdot 18x + 7 \cdot 1040x^2 + 7^6 \cdot 18x^3 + 7^{10}x^4$
11		$1 + 477x + 11 \cdot 16752x^2 + 11^5 \cdot 477x^3 + 11^{10}x^4$
13		$1 + 883x + 13 \cdot 45714x^2 + 13^5 \cdot 883x^3 + 13^{10}x^4$
17		$1 + 426x + 17 \cdot 97368x^2 + 17^5 \cdot 426x^3 + 17^{10}x^4$

In particular, conductor $= N = 2^6 3^5 = 15552$.

Lower bounds for motivic Galois groups via Frobenius elements.

Finite group case. Easy! Let X have dimension zero with $G_X \subseteq S_n$. Lower bounds come from Frobenius partitions

$$\mathrm{Fr}_p \in S_n^\natural = (\text{Partitions of } n).$$

For example, for X coming from $x^9 - x - 1$,

$$(\mathrm{Fr}_2, \mathrm{Fr}_3, \mathrm{Fr}_5) = ((9), (6,3), (5,4)).$$

From Fr_2 and Fr_5 (or Fr_3 and Fr_5), $G_X = S_9$.

Connected group case. Even easier! Suppose M has $G_M \subseteq G$ where G is connected. For "almost any" pair of conjugacy classes c_1, $c_2 \in G^\natural(\mathbb{Q})$, the only subgroup $H \subseteq G$ containing these classes is G itself.

For $M = H([3^3], [2^6], 1)$, take any distinct

$$p_1, p_2 \in \{5, 7, 11, \ldots, 991, 997\}.$$

Then $f_{M,p_1}(x)$ and $f_{M,p_2}(x)$ suffice to prove $G_M = CSp_4$.

Equidistribution of $\mathrm{Fr}_p/p^{w/2} \in G_M^{1\natural}(\mathbb{R})$.

Finite group case. Equidistribution known by Chebotarev density. Example of $x^9 - x - 1$ and first 1000 good primes:

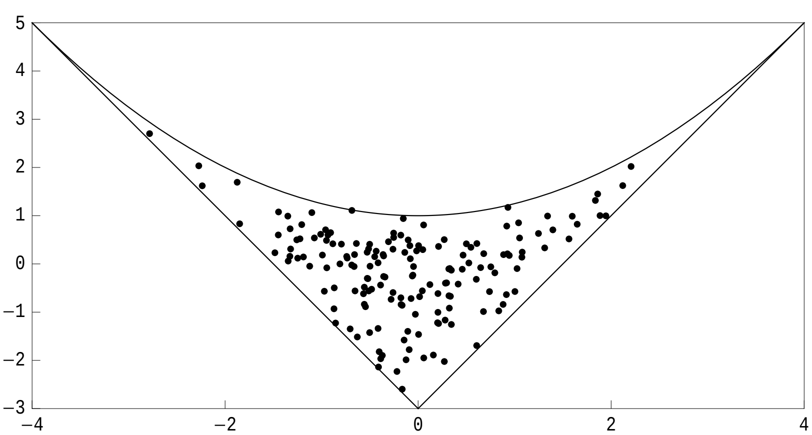
Class	81	9	621	72	711	531	$\cdots$
Freq	$\frac{1.08}{8}$	$\frac{1.01}{9}$	$\frac{0.83}{12}$	$\frac{1.05}{14}$	$\frac{1.18}{14}$	$\frac{1.05}{15}$	$\cdots$

Numerators will be 1 in the limit.

Connected group case. The Sato-Tate conjecture predicts equidistribution. For $Sp_4^\natural$ and coordinates $(t, u) = (\chi_4, \chi_5)$ the density is

$$\frac{1}{4\pi^2}\sqrt{(t^2 - 4u + 4)(u - 2t + 3)(u + 2t + 3)}.$$

Points are $\mathrm{Fr}_p/p^{5/2}$ from $M = H([3^3], [2^6], 1)$:



Responses to the desiderata.

For factorization it's best to extend coefficients from $\mathbb{Q}$ to

$$\mathbb{Q}^{\mathrm{cm}} = (\text{Union of all CM fields in } \mathbb{C}).$$

Corresponding to a decomposition into irreducibles, $H^w(X(\mathbb{C}), \mathbb{Q}^{\mathrm{cm}}) = \bigoplus_M M$, one has a factorization

$$L_S((X, w), s) = \prod_M L_S(M, s),$$

with each $L_S(M, s)$ having coefficients in $\mathbb{Z}^{\mathrm{cm}}$.

A (bad factors $L_p(M, s)$) and B (conductors p^{c_p}). The theory of motivic decomposition groups $\tilde{D}_p$ gives both.

C (Infinite factors $L_\infty(M, s)$). The action of $\tilde{D}_\infty$ likewise gives $L_\infty(M, s)$.

D (Analytic Properties). As M ranges over irreducible motives, the $L(M, s)$ are expected to range over all irreducible automorphic L-functions of algebraic type. Moreover, the Tate conjecture says that this surjection is bijective.

For a given M, one can collect very strong numerical evidence that the expected analytic continuation and functional equation hold (e.g. by *Magma*'s `CheckFunctionalEquation`).

E (Equidistribution). ℓ-adic distribution of $f_{M,p}(x)$ is governed by the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ inside G_M. Archimedean distribution is conjecturally governed by G_M itself.

F (Special values). More of the theory of motives enters. E.g. for $M = H([3^3], [2^6], 1)$, numerically $L(M, 3) = L'(M, 3) = 0$ and

$$L''(M, 3) = 12.6191334778913437117846768\,.$$

So there should be two null-homologous surfaces on any 5-fold underlying M, with $L''(M, 3)$ the product of a period and a regulator.