A database of number fields

David P. Roberts
University of Minnesota, Morris

Work is joint with John Jones of Arizona State University.

The database is an early hit on the Internet search `database number fields`. It pays close attention to the ramification of primes, and is focused on completeness results in degrees ≤ 11. An overall goal is to get a practical feel for the set of all number fields by looking very systematically at first examples.

Our paper describing the database and its interactions with theoretical issues is a later hit and at `arXiv:1404.0266`.

This talk follows the section structure of the paper. On most slides, we pause to actually query the database.

2. Using the database.

Asking for quartic fields with $|D| \leq 250$ returns the complete list of six fields:

Results below are proven complete					
rd(K)	grd(K)	D	h	G	Polynomial
3.29	6.24	$-^{2}3^{2}13^{1}$	1	D_4	$x^4 - x^3 - x^2 + x + 1$
3.34	3.34	$-^{2}5^{3}$	1	C_4	$x^4 - x^3 + x^2 - x + 1$
3.46	3.46	$-^{2}2^{4}3^{2}$	1	V_4	$x^4 - x^2 + 1$
3.71	6.03	$-^{2}3^{3}7^{1}$	1	D_4	$x^4 - x^3 + 2x + 1$
3.87	3.87	$-^{2}3^{2}5^{2}$	1	V_4	$x^4 - x^3 + 2x^2 + x + 1$
3.89	15.13	$-^{2}229^{1}$	1	S_4	$x^4 - x + 1$

The $-^{s}$ in D indicates s complex places.

Asking for quartic fields with discriminant $-^{*}2^{*}3^{*}$ returns all 62 fields.

Clicking on a prime p in e.g. $D = -^{1}2^{6}3^{3}$ for $K = \mathbb{Q}[x]/(x^4 - 3x^2 + 3)$ links into our earlier local field database and returns a thorough description of the completion K_p.

Clicking on e.g. grd = 6.45 gives details behind the Galois Root Discriminant, i.e. the root discriminant of the Galois closure.

4. Summarizing Tables. The paper has one table for each degree ≤ 11. The sextic table:

Degree 6									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\lvert\mathcal{K}[G,\Omega]\rvert$	Tot
1	6	7	0	3	15	5.06	5.06	399	*5291*
2	S_3	8	1	5	31	4.80	4.80	610	*8353*
3	D_6	48	6	10	434	4.93	8.06	3590	*147965*
4	A_4	1	0	0	1	7.32	10.35	59	*1357*
5	$3 \wr 2$	8	0	5	31	4.62	10.06	254	*2169*
6	$2 \wr 3$	7	0	0	15	5.61	12.31	243	*62484*
7	S_4^+	22	3	1	143	5.69	13.56	527	*242007*
8	S_4	22	3	1	143	6.63	13.56	527	*18738*
9	S_3^2	22	0	4	375	7.89	15.53	445	*9721*
10	$3^2:4$	4	0	2	44	8.98	23.57	34	*396*
11	$2 \wr S_3$	132	18	2	2002	4.65	16.13	2196	*323148*
12	$PSL_2(5)$	0	5	6	62	8.12	18.70	78	*275*
13	$3^2:D_4$	50	0	0	624	4.76	21.76	274	*27049*
14	$PGL_2(5)$	5	38	22	1353	11.01	24.18	192	*11519*
15	A_6	8	2	4	*540*	8.12	31.66	10	*670*
16	S_6	54	30	42	*8334*	4.95	33.50	26	*21594*

Regular type indicates a completeness result. Thus for S_6 sextic fields:
There are exactly 54 ramified within $\{2,3\}$ and at least 8334 ramified within $\{2,3,5\}$. The smallest rd is $14731^{1/6} \approx 4.95$ while the smallest grd is $2^{9/4}3^{4/5}5^{2/3} \approx 33.50$. There are 26 fields with grd $\leq \Omega = 8\pi e^{\gamma} \approx 44.76$ and currently 21594 fields overall on the database.

5. S_5 **quintics with discriminant** $-^*2^*3^*5^*7^*$**.**

There are 11279 of them (determined by a huge search!).

The distribution of these fields by discriminant reflects the distribution of local algebras in accordance with general mass heuristic principles (Bhargava, Malle).

For example, the total mass of quintic 2-adic algebras of discriminant 2^a is on the second row:

a	0	1	2	3	4	5	6	7	8	9	10	11
$/\mathbb{Q}_2$	1		2	2	5	4	6		4	4	4	8
$/\mathbb{Q}$	205		468	416	1327	1081	1597		1260	1233	1171	2521

Queries give the numbers on the third row of quintic S_5 fields with discriminants $-^*2^a3^*5^*7^*$.

6. Low degree nonsolvable fields with discriminant $-^*p^*q^*$.

The case of septics (and some octics) illustrates the boundary of computational feasibility:

$SL_3(2)$ and $PGL_2(7)$

	2	3	5	7	11	13
2	•	*4*		*51*		
3	0	•		*28*		
5	0	0	•	*4*		
7	*44*	*12*	*4*	•	*4*	*6*
11	4	0		*6*	•	
13	0				0	•

A_7 and S_7

	2	3	5	7	11	13
2	•	10	24	*55*	0	0
3	0	•	14	*44*	2	
5	2	3	•	*18*		
7		*7*	*5*	•	*5*	
11	0	0	*1*		•	0
13	0				0	•

Queries for $-^*p^*q^*$ septics give more details (Note: group-theoretic information is available by clicking).

7. Nilpotent octic fields with odd discriminant $-^*p^*q^*$.

For each set $\{p, q\}$ of two odd primes, there is a group $G_{p,q} = \langle \tau_p, \tau_q \rangle$ governing number fields with discriminant $-^*p^*q^*$ and Galois group of order 2^*. The groups $G_{p,q}$ have been studied by Boston, Ellenberg, and others. Some are finite, others are infinite.

The database has all cases with p, $q < 250$. There are 23 possibilities for the quotient of $G_{p,q}$ governing octic fields. In the representative case with $p \equiv q \equiv 5$ (8), there are 4 possibilities:

		Number of fields with a given Galois group $8Tj$																
p	q	1	2	4	5	6	7	8	10	16	17	19	20	21	27	28	30	Freq
5	13		3				2											1/32
5	29		3	3	1				6		8	4	2	2	4	4		1/128
13	53		3	3	1	2		2	6		8	12	6	6	12	12	16	1/128
13	29		3	3	1				6	2	12	4	2	2	2	2	4	1/64

Queries show that e.g. $\{5, 181\}$ is in the case represented by $\{13, 53\}$ at our current octic level. But they also show that 2-primary parts of class numbers disagree, so $G_{13,53} \neq G_{5,181}$.

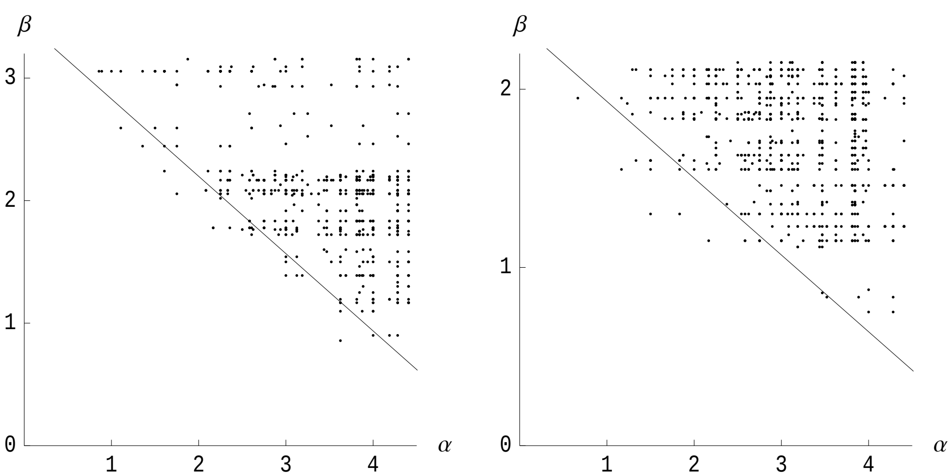
8. Nilpotent octic fields with discriminant $-^*2^*q^*$.

Cases with $p = 2$ are greatly complicated by the fact that ramification at 2 is wild. The database covers $q < 2500$ where there are 13 possibilities for the quotient of $G_{p,q}$ governing octic fields. The two cases with the smallest number of fields are $q \equiv 3, 5$ (8), where $G_{2,q}$ in its entirety is known (Koch) to be the pro-2 free product $D_\infty * D_q$.

	$\|G\| = 8$					$\|G\| = 16$						$\|G\| = 32$					$\|G\| = 64$					
																21						
																20		28^2	31^2			
q	1	2	3	4	5	6^2	7	8	9^4	10^2	11^3	15^2	16^2	17^2	18^8	19^2	26^4	27^2	29^6	30^4	35^8	Tot
$\mathbb{Q}_2$	24	18	1	18	6	16	36	36	9	12	16	38	12	48	4	24	24	48	16	24	48	1449
3	4	6	1	14	2	10	6	22	7	4	6	21	4	8	3	4	16	8	8	4	21	579
5	8	18	1	12	0	10	20	10	6	12	6	21	12	36	1	12	6	24	4	2	9	621

It is also known (Koch) that $G_{2,q}$ coincides with its 2-decomposition group when $q \equiv 3, 5$ (8). From the database, we observe that the 579 or 621 octic 2-adic fields are in fact independent of q! Some of this may be seen directly by queries (with $q \equiv 3$ (8) and $\text{ord}_2(D) = 16$ a manageably small case).

9. Minimal nonsolvable fields with grd $\leq \Omega$. Fields with small grd are of particular note: they interact interestingly with Serre-Odlyzko GRH analytic bounds and corresponding automorphic forms can often be found. All grd's $2^\alpha 3^\beta$ and $2^\alpha 5^\beta$ coming from minimal nonsolvable fields on the $n \leq 11$ part of the database:



It is hard to keep grd's under Ω. For example,

$$\begin{aligned} f_1(x) &= x^5 - 10x^3 - 20x^2 + 110x + 116, \\ f_2(x) &= x^5 + 10x^3 - 10x^2 + 35x - 18, \\ f_3(x) &= x^5 + 10x^3 - 40x^2 + 60x - 32 \end{aligned}$$

all have $G = A_5$ and grd $= 2^{3/2}5^{8/5} \approx 37.14$. But their pairwise products all have grd $>> \Omega$.

10. General nonsolvable fields with grd $\leq \Omega$. Up through now, we have been mainly counting collections of number fields with given properties. But individual number fields can be of interest! For example, searching for the grd $1831^{1/2} \approx 42.79$ returns five interrelated polynomials, including

$$\begin{aligned} f_1(x) &= x^{11} - 2x^{10} + x^9 - 5x^8 + 13x^7 \\ &\quad -9x^6 + x^5 - 8x^4 + 9x^3 - 3x^2 - 2x + 1, \end{aligned}$$

$$\begin{aligned} f_2(x) &= x^{19} - 6x^{18} + 18x^{17} - 39x^{16} + 73x^{15} \\ &\quad -200x^{14} + 265x^{13} + 305x^{12} - 931x^{11} \\ &\quad +1905x^{10} - 5214x^9 + 10284x^8 \\ &\quad -13343x^7 + 12719x^6 - 8662x^5 \\ &\quad +4443x^4 - 1732x^3 + 614x^2 - 152x + 39. \end{aligned}$$

The splitting field K_1 is by far the least ramified $PSL_2(\mathbb{F}_{11})$ field known. The splitting field K_2 has Galois group D_{19}, being the Hilbert class field of $\mathbb{Q}(\sqrt{-1831})$. The compositum K_1K_2 has degree $660 \cdot 38 = 25080$, which is quite large for its grd of ≈ 42.79. One can build other similarly remarkable fields by carefully combining fields on the database.