A THREE-PARAMETER CLAN OF HURWITZ-BELYI MAPS

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Abstract. We study a collection of Hurwitz-Belyi maps depending on three integer parameters, finding formulas uniform in the parameters.

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1. Introduction

This paper is a companion to *Hurwitz-Belyi maps* [6]. It makes use of some of the terminology and notation set up in the first three sections there, but is otherwise self-contained. The main fact a reader familiar with Belyi maps has to know from [6] is that Hurwitz-Belyi maps are a particularly interesting type of Belyi map that arise in moduli problems.

A well-known phenomenon is that certain infinite collections of Belyi maps can be profitably studied simultaneously by means of parameters. We informally refer to such a collection as a *clan*. For example, the recent papers [4],[5] study Belyi maps which are uniquely determined by partition triples $(\lambda_0, \lambda_1, \lambda_\infty)$, with λ_∞ of the form $(m, 1, \ldots, 1)$. These papers find ten clans and ten sporadic examples.

Twenty years ago in [1], Couveignes found what in our language we call a four-parameter clan of Hurwitz-Belyi maps. To our knowledge, it is the only such clan systematically studied in the literature. However we are convinced from preliminary computations that there are many other natural clans of Hurwitz-Belyi maps.

Our purpose in this paper is to call attention to the mostly unexplored topic of clans of Hurwitz-Belyi maps. We aim to indicate the general nature of all clans by studying the Couveignes clan further. Naturally, we focus on aspects of this clan which are not simple consequences of the results of [1].

While the study of Belyi maps always has a very number-theoretic feel, the study of clans of Belyi maps also brings in many notions from the theory of special functions. For example, Jacobi polynomials, Padé approximants, and differential equations are all prominent in [5]. Unlike the cases of [5], a Hurwitz-Belyi map in a clan is typically not determined by its partition triple. As a consequence, the special functions from Hurwitz-Belyi maps can be quite far removed from classical special functions.

Section 2 sets the stage by reviewing some of Couveignes' results. Sections 3-7 pursue topics that would be natural to study in any clan. Section 3 first reduces to one fewer parameter to simplify this whole paper, defining Hurwitz-Belyi maps

$$\pi_{a,b,c} : \mathsf{X}_{a,b,c} \to \mathsf{P}^1. \tag{1.1}$$

of degree $m = 3(a + b + 2c)$. It then gives a uniform algebraic description of these maps. Section 4 describes the monodromy group of $\pi_{a,b,c}$ in S_m, identifying exactly when it is primitive. Section 5 obtains a discriminant formula from which one obtains the exact set of primes at which $\pi_{a,b,c}$ has bad reduction. Section 6 discusses wall-crossing phenomena which arise naturally when studying clans. Section 7 compares the Hurwitz-Belyi maps (1.1) with other Belyi maps sharing the same ramification partitions. Some of the more complicated formulas in this paper are available in the *Mathematica* file `TPC.m` on the author's homepage.

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2. Couveignes' cubical clan

In this section we define Couveignes' clan, using notation adapted to our context, and present some of his results.

2.1. **Direct description.** Let a, b, c, and d be distinct positive integers and set $n = a + b + c + d$. Consider maps F from the complex projective y-line P^1_y to the complex projective t-line P^1_t given by

$$t = F(y) = (1 - x_1 y)^a (1 - x_2 y)^b (1 - x_3 y)^c (1 - x_4 y)^d. \tag{2.1}$$

Here the x_i are currently unspecificed distinct complex numbers. Clearly, the preimage of $0 \in \mathsf{P}^1_t$ consists of the points $1/x_1$, $1/x_2$, $1/x_3$, $1/x_4$ in P^1_y of respective multiplicities a, b, c, and d. Accordingly, the ramification partition for 0 is $\lambda_0 = (a, b, c, d)$. Likewise, the preimage of $\infty \in \mathsf{P}^1_t$ is simply $\infty \in \mathsf{P}^1_y$, giving the ramification partition $\lambda_\infty = (n)$. Note that $F(0) = 1$. Couveignes' starting point is to require also $F'(0) = F''(0) = 0$. Explicitly this requirement translates to the following elegant conditions on the x_i from [1, §5.1]:

$$ax_1 + bx_2 + cx_3 + dx_4 = 0, \tag{2.2}$$
$$ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 = 0. \tag{2.3}$$

The point $0 \in \mathsf{P}^1_y$, always in the preimage of $1 \in \mathsf{P}_t$, now has multiplicity 3. For generic $(x_1, \ldots, x_4)$ satisfying (2.2) and (2.3), the ramification partition λ_1 is $(3, 1, \ldots, 1)$. There is then a single remaining critical point y_{crit} on P^1_y. It maps to some point $v \in \mathsf{P}^1_t$, and the ramification partition for v is $\lambda_v = (2, 1, \ldots, 1)$. The isomorphism type of F depends only on $(x_1, \ldots, x_4)$ in the projective curve $\mathsf{X}_{a,b,c,d}$ defined by (2.2) and (2.3). There is a natural map

$$\pi_{a,b,c,d} : \mathsf{X}_{a,b,c,d} \to \mathsf{P}^1_v. \tag{2.4}$$

Namely a point $(x_1, x_2, x_3, x_4) \in \mathsf{X}_{a,b,c,d}$ is taken into its unique extra critical value v.

2.2. **Hurwitz parameter.** The previous paragraph fits into the formalism of [6, §3.1] as follows. The Hurwitz parameter describing the desired functions F is

$$h(a,b,c,d) = (S_n,\ (2\,1^{n-2},\ a\,b\,c\,d,\ 3\,1^{n-3},\ n),\ (1,\ 1,\ 1,\ 1)).$$

Attention is thoroughly focused at the moment on the the local ramification partitions $(\lambda_v, \lambda_0, \lambda_1, \lambda_\infty)$. The multiplicity vector $(1,1,1,1)$ will remain trivial even in our modifications below. While we only consider F if its global monodromy group is all of S_n, this condition turns out to be forced by the local ramification partitions. The maps (2.4), arising as they do from a moduli problem, are Hurwitz-Belyi maps.

2.3. **Braid monodromy and degree.** As a Belyi map, $\pi_{a,b,c,d}$ has its own ramification partitions. We call them β_0, β_1, and β_∞, to distinguish them from the earlier λ_t. This notation is used throughout [6], with the letter β being a reminder that partition triples of Hurwitz-Belyi maps can always be computed by braids.

The braid ramification partitions are are deducible from the figure in [1, §5.2], being

$$\begin{aligned} \beta_0 &= (a+b)^2(a+c)^2(a+d)^2(b+c)^2(b+d)^2(c+d)^2, \\ \beta_1 &= 4^6 1^{6n-24}, \\ \beta_\infty &= (a+b+c)^2(a+b+d)^2(a+c+d)^2(b+c+d)^2. \end{aligned} \tag{2.5}$$

In particular the degree of $\pi_{a,b,c,d}$ is $m = 6n$.

The total number of parts in (2.5) is $12 + (6 + m - 24) + 8 = m + 2$, and so $\mathsf{X}_{a,b,c,d}$ has genus zero. For other clans of Hurwitz-Belyi maps, the degree can grow faster than linearly in the parameters, unlike $m = 6(a+b+c+d)$. The genus of the covering curves can be larger than 0, unlike the genus of $\mathsf{X}_{a,b,c,d}$. In these senses, the Couveignes clan is particularly simple. naturall

In the current context, it is better to modify the standard visualization conventions of [6, §2.3], to exploit that the number of parts in β_0 and β_∞ is small and independent of the parameters. Accordingly, we now view the interval $[-\infty, 0]$ in the projective line P^1_v as the simple bipartite graph •—∘. The dessin $\pi^{-1}_{a,b,c,d}([-\infty,0]) \subset \mathsf{X}_{a,b,c,d}$, capturing Couveignes' determination [1, §5.2 and §9] of the permutation triple (b_0, b_1, b_∞) underlying $(\beta_0, \beta_1, \beta_\infty)$, is indicated schematically by Figure 2.1. Note that our visualization is dual to that of Couveignes, as our dessin is formatted on a cube rather than an octahedron.

2.4. **Failure of rationality.** The $\mathbb{Q}$-curve $\mathrm{X}_{a,b,c,d}$ underlying the Riemann surface $\mathsf{X}_{a,b,c,d} = \mathrm{X}_{a,b,c,d}(\mathbb{C})$ is naturally given in the projective space P^3 by the system (2.2), (2.3). The second equation has no solution in $\mathrm{P}^3(\mathbb{R})$ and so $\mathrm{X}_{a,b,c,d}(\mathbb{R})$ is empty. This non-splitting of X over $\mathbb{R}$, which forces non-splitting over $\mathbb{Q}_p$ for an odd number of primes p, is one of the main focal points of [1].

2.5. **More general parameters.** Note finally that our requirement that a, b, c, and d are all distinct is just so that the above considerations fit immediately into the formalism of Hurwitz-Belyi maps. One actually has natural covers $\mathrm{X}_{a,b,c,d} \to \mathrm{P}^1$ of degree $6n$ even when this requirement is dropped. These covers have extra symmetries, as illustrated by the rotation ι discussed in the next section. Couveignes

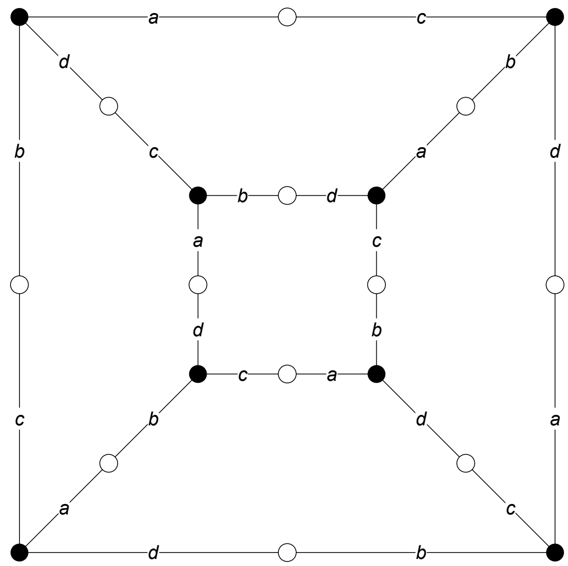


Figure 2.1. Schematic indication of Couveignes' dessin with parameters (a, b, c, d) based on the combinatorics of a cube. The actual dessin is obtained by replacing each $\bullet\!\!-\!u\!-\!\!\circ$ by u parallel edges.

allowed the parameters to become zero and negative, and we will do the same in Section 6.

3. The semicubical clan

Couveignes does not explicitly give the map $X_{a,b,c,d} \to P^1_v$. The map cannot be given simply by a rational function in $\mathbb{Q}(x)$, because, as just discussed in §2.4, the curve $X_{a,b,c,d}$ is not isomorphic to P^1 over $\mathbb{Q}$. The fact that all multiplicities are even in the triple (2.5) is necessary for this somewhat rare obstruction.

In this section, we simplify by modifying the situation so that the covering curves become isomorphic to P^1 over $\mathbb{Q}$. Then we give corresponding rational functions.

3.1. **Restriction to three parameters.** For our modified clan, we still require that a, b, and d are distinct. But now we essentially set $c = c$ in Couveignes' situation, so that the degree takes the asymmetric form $n = a + b + 2c$. We thus are now considering the Hurwitz parameters

$$h(a, b, c) = (S_n, (2\,1^{n-2},\ a_0\,b_1\,c^2,\ 3_x\,1^{n-3},\ n_\infty), (1, 1, 1, 1)). \tag{3.1}$$

The four subscripts are present for the purposes of normalization and coordinatization. They will enter into our proof of Theorem 3.1 below.

The fact that two distinguishable points have now become indistinguishable implies that $\mathsf{X}_{a,b,c} = \mathsf{X}_{a,b,c,c}/\iota$, where ι is the rotation interchanging c and d in Figure 2.1. The fixed points of this rotation are the upper-left and lower-right white vertices, each with valence $c + d = 2c$. Thus $\mathsf{X}_{a,b,c}$ has degree $3n$ over P^1_v. The new braid partition triple is

$$\beta_0 \quad = \quad (a+b)_\infty (a+c)^2 (b+c)^2 c^2,$$

$$\begin{aligned} \beta_1 &= 4^3 1^{3n-12}, \\ \beta_\infty &= (a+b+c)^2(a+2c)_0(b+2c)_1. \end{aligned} \tag{3.2}$$

There are three singletons, namely the parts subscripted 0, 1, and ∞. So not only is the $\mathbb{Q}$-curve $X_{a,b,c}$ split, but also our choice of subscripts gives it a canonical coordinate. Because of the equation $\mathsf{X}_{a,b,c} = \mathsf{X}_{a,b,c,c}/\iota$, we call the clan indexed by $h(a,b,d)$ the *semicubical clan.*

3.2. **Explicit rational functions.** To compute $\pi_{a,b,c}$ for given integers a, b, c as an explicit rational function, we follow the standard procedure illustrated by simple examples in Sections 2 and 4 of [6]. Remarkably, this computation can be done for all a, b, and c at once:

Theorem 3.1. *For distinct positive integers a, b, c, let $n = a+b+2c$ and*

$$\begin{aligned} A &= -2nx(a+c) + (a+c)(a+2c) + nx^2(n-c), \\ B &= a(a+c) - 2anx + nx^2(-(c-n)), \\ C &= x^2(a+b)(n-c) - 2ax(n-c) + a(a+c), \\ D &= nx^2(a+b) + a(a+2c) - 2anx. \end{aligned}$$

Then the Hurwitz-Belyi map for (3.1) *is*

$$\pi_{a,b,c}(x) = \frac{a^a b^b A^{a+c} B^{b+c} D^c}{2^c c^{2c} n^n x^{a+2c} (1-x)^{b+2c} C^{n-c}}. \tag{3.3}$$

Proof. The polynomial

$$F_x(y) = \frac{y^a (y-1)^b (y^2+ry+s)^c}{x^a (x-1)^b (x^2+rx+s)^c} \tag{3.4}$$

partially conforms to (3.1), including that $F_x(x) = 1$. From the 3_x we need also that $F_x'(x) = 0$ and $F_x''(x) = 0$. The derivative condition is satisfied exactly when

$$r = \frac{-nx^3 + (a+2c)x^2 - (a+b)sx + as}{((a+b+c)x^2 - (a+c)x)}.$$

The second derivative condition is satisfied exactly when

$$s = \frac{n(a+b+c)x^4 - 2n(a+c)x^3 + (a+c)(a+2c)x^2}{(a+b)(a+b+c)x^2 - 2a(a+b+c)x + a(a+c)}.$$

The identification of r and s completely determine the maps $F_x : \mathsf{P}^1_y \to \mathsf{P}^1_t$.

From a linear factor in the numerator of $F_x'(y)$, one gets that the critical point corresponding to the 2 in the first class $2\,1^{n-2}$ in (3.1) is

$$y_x = \frac{as}{nx^2}.$$

Substantially simplifying $F_x(y_x)$ gives the right side of (3.3). □

From the form of the normalized partition tuple (3.2), one knows *a priori* that

$$\pi_{a,b,c}(1-x) = \pi_{b,a,c}(x). \tag{3.5}$$

Indeed, one can check that the simultaneous interchange $a \leftrightarrow b$, $x \leftrightarrow 1-x$ interchanges A and B and fixes C and D. Given this fact, the symmetry (3.5) is visible in the main formula (3.3).

3.3. **Dessins.** As with Couveignes' cubical clan, the semicubical clan gives covers $\pi_{a,b,c}$ even when a, b, and c are not required to be distinct. For example, the simplest case is the dodecic cover

$$\pi_{1,1,1}(x) = -\frac{\left(6x^2-8x+3\right)^2\left(8x^2-8x+3\right)\left(6x^2-4x+1\right)^2}{2^8x^3(x-1)^3\left(3x^2-3x+1\right)^3}.$$

An example representing the main case of distinct parameters is

$$\pi_{7,6,4}(x) = \frac{\left(119x^2-154x+55\right)^{11}\left(13x^2-14x+5\right)^4\left(51x^2-42x+11\right)^{10}}{2^{14}x^{15}(x-1)^{14}\left(221x^2-238x+77\right)^{17}}.$$

Let $\gamma(a,b,c) = \pi^{-1}_{a,b,c}([-\infty,0]) \subset \mathsf{P}^1_x$. The left part of the Figure 3.1 is a view on $\gamma(1,1,1)$. To obtain the general $\gamma(a,b,c)$ topologically, one replaces each segment of $\gamma(1,1,1)$ by the appropriate number a, b, or c of parallel segments, so as to create $m = 3n = 3a+3b+6c$ edges in total. As an example, the right part of the figure draws the degree sixty-three dessin $\gamma(7,6,4)$.

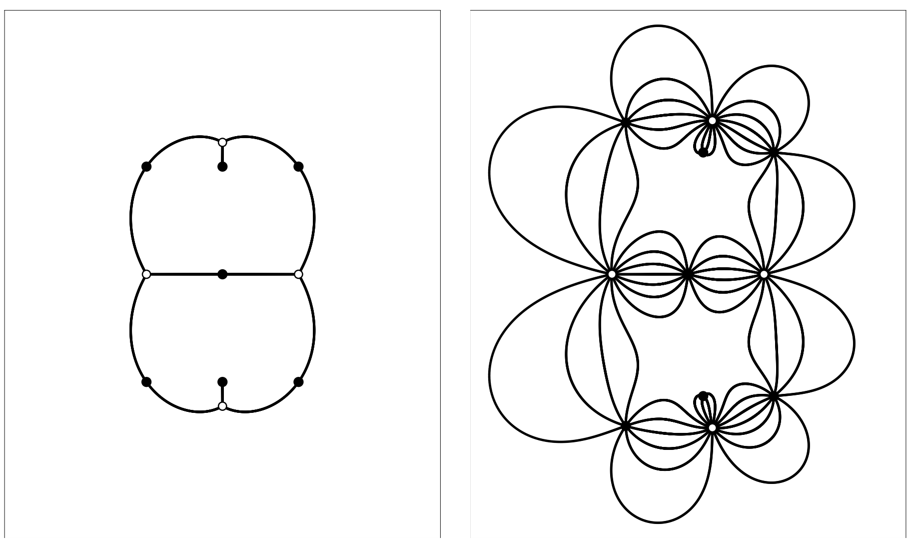


FIGURE 3.1. Left: $\gamma(1,1,1)$. Right: $\gamma(7,6,4)$.

The view on $\gamma(a,b,c)$ given by (3.1) was obtained via the involution $s(x) = x/(2x-1)$ which fixes 0 and 1 and interchanges ∞ and $1/2$. The sequence of white, black, and white vertices on the real axis are the points 0, ∞, and 1 in the Riemann sphere. Thus the point not in the plane of the paper is the point $x = 1/2$. The involution (3.5) corresponds to rotating the figure one half-turn about its central point ∞.

4. Monodromy

A degree m Belyi map $\mathsf{X} \to \mathsf{P}^1$ has a monodromy group, which is a subgroup of S_m well-defined up to conjugation. A natural question is to determine this monodromy group. This section essentially answers this question for members of the semicubical clan, and also for all other Belyi maps sharing the same ramification partition triple.

4.1. **Three imprimitive cases.** Suppose in general that X is connected, as all our $\mathsf{X}_{a,b,c}$ are. Group-theoretically, we are supposing that the monodromy group is transitive. Then a natural first question is whether $\mathsf{X} \to \mathsf{P}^1$ has strictly intermediate covers. As preparation for the next subsection, we exhibit three settings where there is such an intermediate cover

(4.1) $$\mathsf{X}_{a,b,c} \stackrel{\delta}{\to} \mathsf{Y} \stackrel{\epsilon}{\to} \mathsf{P}^1_v.$$

The monodromy group is imprimitive exactly when there exists a Y as in (4.1), and primitive otherwise.

Case 1. Let $e = \gcd(a,b,c)$. If $e > 1$ then the explicit formula (3.3) says that

(4.2) $$\pi_{a,b,c}(x) = \pi_{a/e,b/e,c/e}(x)^e.$$

Thus one has imprimitivity here, with the cover ϵ naturally coordinatized to $y \mapsto y^e$.

Case 2. Suppose $a = b$. As a special case of (3.5), the cover $\pi_{a,a,c}$ has the automorphism $x \mapsto 1-x$, corresponding to rotating dessins as in Figure 3.1. To coordinatize Y, we introduce the function $y = \delta(x) = x(1-x)$. Then $\epsilon(y)$ works out to

(4.3)
$$\pi^V_{a,c}(y) = \frac{(4ay - a + 4cy - 2c)^c \left(4y^2(2a+c)^2 - 4ay(2a+3c) + a(a+2c)\right)^{a+c}}{c^{2c} 2^{2a+3c} y^{a+2c} (4ay - a + 2cy - c)^{2a+c}}.$$

The superscript V indicates that $\pi^V_{a,c}$ comes from $\pi_{a,a,c,c}$ by quotienting by a noncyclic group of order four. The ramification partitions and the induced normalization of $\pi^V_{a,c}$ are

(4.4)
$$\begin{aligned} \alpha_0 &= a_\infty \, c\,(a+c)^2, \\ \alpha_1 &= 4\,2_4\,1^{1.5n-6}, \\ \alpha_\infty &= (a+2c)_0,\,(2a+c). \end{aligned}$$

The covers $\pi^V_{a,c}$ and $\pi^V_{c,a}$ are isomorphic, although our choice of normalization obscures this symmetry.

Case 3. Suppose $c \in \{a,b\}$. Via (3.5), it suffices to consider the case $b = c$. Then while the cover $\pi_{a,c,c}$ does not have any automorphisms, the original cubical cover $\pi_{a,c,c,c}$ has automorphism group S_3, This implies that $\pi_{a,c,c}$ has a subcover $\epsilon = \pi^S_{a,c}$ of index three. To coordinatize Y in this case, we use the function

$$y = \delta(x) = \frac{(x-1)^3(a+c)(a+2c)}{x\left(x^2(a+c)(a+2c) - 2ax(a+2c) + a(a+c)\right)}.$$

Then
(4.5)
$$\pi^S_{a,c}(y) = \frac{(-a)^a (y-1)^{a+c} \left(a^3y^2(a+c) + 2a^2cy(5a+9c) + 27c^2(a+c)(a+2c)\right)^c}{2^c c^c n^n y^c}.$$

So the ramification partitions and the induced normalization of $\pi^S_{a,c}$ are

(4.6)
$$\begin{aligned} \alpha_0 &= (a+c)_1\, c^2 \\ \alpha_1 &= 4\,1^{n-4}, \end{aligned}$$

$$\alpha_\infty = (a+2c)_\infty\, c_0.$$

4.2. **Primitivity.** The next theorem says in particular that all $\pi_{a,b,c}$ not falling into Cases 1-3 of the previous subsection have primitive monodromy.

Theorem 4.1. *Let a, b, c be distinct positive integers with* $\gcd(a,b,c)=1$. *Let $\pi : \mathsf{X} \to \mathsf{P}^1$ be a Belyi map with the same ramification partition triple* (3.2) *as $\pi_{a,b,c}$. Then π has primitive monodromy.*

Note that the hypothesis $\gcd(a,b,c)=1$ excludes Case 1 from the previous subsection. The distinctness hypothesis excludes Cases 2 and 3. Since these cases all have imprimitive monodromy, we will have to use these hypotheses.

Proof. The hypothesis $\gcd(a,b,c)=1$ has implications on the parts of β_0 and β_∞ in (3.2). For β_0 it implies $\gcd(a+b,a+c,b+c,c)=1$ while for β_∞ it implies $\gcd(a+b+c,a+2c,b+2c)\in\{1,3\}$. As before, let $n=a+b+2c$ and $m=3n$. The two hypotheses together say that the smallest possible degree m is twenty-one, coming from $(a,b,c)=(3,2,1)$.

Let Y be a strictly intermediate cover as in (4.1), with $\mathsf{X}_{a,b,c}$ replaced by X. Let e be the degree of $\mathsf{Y}\to\mathsf{P}^1$ and write its ramification partition triple as $(\alpha_0,\alpha_1,\alpha_\infty)$. Because $\gcd(a+b,a+c,b+c,c)=1$, the cover Y cannot be totally ramified over 0. Because $\gcd(a+b+c,a+2c,b+2c)\in\{1,3\}$, it can be totally ramified over infinity only if $e=3$. There are then two possibilities, as $(\alpha_0,\alpha_1,\alpha_\infty)$ could be $((1,1,1),(3),(3))$ or $((2,1),(2,1),(3))$. The $\alpha_1=3$ in the first possibility immediately contradicts $\beta_1=(4^3,1^{m-12})$. The $\alpha_1=(2,1)$ in the second possibility allows two possible forms for β_1, namely $(4^3,1^6)$ and (4^3). But both of these have degree less than twenty-one. So $e=3$ is not possible, and thus Y cannot be totally ramified over ∞ either.

Since α_0 and α_∞ both have at least two parts, the minimally ramified partition $(2,1^{e-2})$ is eliminated as a possibility for α_1, by the Riemann-Hurwitz formula. The candidates $(2^2,1^{e-4})$ and $(2^3,1^{e-6})$ for α_1 both force X to be a double cover of Y, so that $e=m/2$; but both are then incompatible with $\beta_1=(4^3,1^{m-12})$. This leaves $(4,2,1^{m/2-6})$ and $(4,1^{m/3-4})$ as the only possibilities for α_1. In the first case, the two critical values of the double cover $\mathsf{X}\to\mathsf{Y}$ would have to correspond to the 2 in α_1 and the image of the singleton $a+b$ of β_0; the parts of β_∞ would have to be those of α_∞ with multiplicities doubled; from the form of β_∞ in (3.2) this forces $a=b$, putting us in Case 2 and contradicting the distinctness hypothesis. In the second case, the combined partition $\alpha_0\alpha_\infty$ would have the form (k_1,k_2,k_3,k_4,k_5) and the the combined partition $\beta_0\beta_\infty$ would have the form $(3k_1,2k_2,k_2,2k_3,k_3,k_4^3,k_5^3)$ or $(3k_1,3k_2,k_3^3,k_4^3,k_5^3)$. From (3.2), the first possibility occurs exactly when a or b equals c, putting us into Case 3 and contradicting the distinctness hypothesis; the second possibility cannot occur as it is incompatible with the shape of $\beta_0\beta_\infty$ in (3.2). We have now eliminated all possibilities for α_1 and so $\mathsf{X}\to\mathsf{P}^1$ has to be primitive. □

4.3. **Fullness.** In [6] we heavily emphasized full Belyi maps, meaning maps with monodromy group the alternating or symmetric group on the degree m. A natural question is whether *primitive* in Theorem 4.1 can be strengthened to *full*. In the setting of Theorem 4.1, only S_m can appear because $\beta_1=4^3 1^{m-12}$ in (3.2) is an odd partition.

The two smallest degrees of covers as in Theorem 4.1 are $m = 21$ and $m = 24$. There are nine primitive groups in degree 21 and five in degree 24, all accessible via *Magma*'s database of primitive groups. None of them, besides S_{21} and S_{24}, contain an element of cycle type $4^3 1^{m-12}$.

In an e-mail to the author on July 5, 2016, Kay Magaard has sketched a proof that, for all $m \geq 25$, likewise $4^3 1^{m-12}$ is not a cycle partition for a primitive proper subgroup of S_m. Magaard's proof appeals to Theorem 1 of [2], which has as essential hypothesis that the 1's in $4^3 1^{m-12}$ contribute more than half the degree. Special arguments are needed to eliminate the other possibilities that parts 1, 2, and 3 of Theorem 1 of [2] leave open. Thus, Theorem 4.1 can indeed be strengthened by replacing *primitive* by *full*.

5. Primes of bad reduction

A natural problem for any Belyi map defined over $\mathbb{Q}$ is to identify its set $\mathcal{P}$ of primes of bad reduction. In this section, we identify this set for the maps $\pi_{a,b,c}$.

5.1. **Eleven sources of bad reduction.** The dessins $\gamma(a, b, c)$ have four white vertices: 0, 1, and the roots of D. They have seven black vertices: ∞ and the roots of ABC. If any of these eleven points agree modulo a prime p, then the map $\pi_{a,b,c}$ has bad reduction at p. To study bad reduction, one therefore has to consider some special values, discriminants, and resultants. Table 5.1 gives the relevant information, with "value at ∞" meaning the coefficient of x^2 in the quadratic polynomial heading the column.

	A	B	C	D
Value at 0	$(a+c)(a+2c)$	$a(a+c)$	$a(a+c)$	$a(a+2c)$
Value at 1	$b(b+c)$	$(b+c)(b+2c)$	$b(b+c)$	$b(b+2c)$
Value at ∞	$(a+b+c)n$	$(a+b+c)n$	$(a+b)(a+b+c)$	$(a+b)n$
Disc.	$-4bc(a+c)n$	$-4ac(b+c)n$	$-4abc(a+b+c)$	$-8abcn$
Res. with A		$4c^3n^2e$	$4b^2c^3e$	$b^2c^2(a+2c)^2n^2$
Res. with B			$4a^2c^3e$	$a^2c^2(b+2c)^2n^2$
Res. with C				$a^2b^2(a+b)^2c^2$

Table 5.1. Special values, discriminants, and resultants of the four quadratic polynomials A, B, C, D from Theorem 3.1, using the abbreviation $e = (a+c)(b+c)(a+b+c)$.

5.2. **A discriminant formula.** Combining the explicit formula (3.3), the general discriminant formula [3, (7.14)], and the elementary facts collected in Table 5.1, one gets the following discriminant formula.

Corollary 5.1. *Let $a^a b^b A^{a+c} B^{b+c} D^c - v 2^c c^{2c} n^n x^{a+2c} (1-x)^{b+2c} C^{n-c}$ be the polynomial whose vanishing defines $\pi_{a,b,c}$. Its discriminant is*

$$D(a,b,c) =$$
$$(-1)^{(a-1)a/2+(b-1)b/2+c} 2^{n(c+2n)}$$
$$a^{2n^2-a^2+2an-n} b^{2n^2-b^2-n+2bn} c^{(10n^2-(1+a+b)(a+b+3n))/2}$$
$$(a+b)^{(a+b+c-1)n} (a+c)^{an+a+cn+c+n^2-n} (b+c)^{bn+b+cn+c+n^2-n}$$

$$(a+2c)^{(a+2c)^2}(b+2c)^{(b+2c)^2}(a+b+c)^{(a+b+c)(2(a+b+c)+1)}$$
$$n^{n(3n+2)}v^{3n-7}(v-1)^9.$$

In particular, the set $\mathcal{P}_{a,b,c}$ of bad primes of $\pi_{a,b,c}$ is the set of primes dividing

$$abc(a+b)(a+c)(b+c)(a+b+c)(a+2c)(b+2c)(a+b+2c). \qquad \square \tag{5.1}$$

Because of the nature of [3, (7.14)], each factor in the discriminant formula has specific sources on Table 5.1. The discriminants corresponding to $\pi^V_{a,c}$ and $\pi^S_{a,c}$ are given by similar but slightly simpler formulas.

5.3. **Responsiveness to the inverse problem of [6].** Let a, b, c be distinct positive integers without a common factor. The fullness conclusion of §4.3 and the identified prime set (5.1) combine to say that the explicit rational Hurwitz-Belyi maps $\pi_{a,b,c}$ of Theorem 3.1 respond in a uniform way to the inverse problem of §1.3 of [6].

All the clans we have looked at seem to share the property that the analog to $\mathcal{P}_{a,b,c}$ is relatively sparse, but nevertheless grows with the parameters. In [6], we were most interested in fixing a small $\mathcal{P}$ and providing examples of full rational covers in degrees as large as possible. In this direction, clans do not seem to be helpful. The fundamental problem is that in clans the groups G in the Hurwitz parameters are A_n or S_n and one is increasing n. To obtain a sequence establishing Conjecture 11.1 of [6], we expect that instead one has to fix G, and consider moduli problems in which the number of critical values, always four in this paper, tends to infinity.

6. Allowing negative parameters

The formula (3.3) for $\pi_{a,b,c}$ makes sense for arbitrary integer parameters satisfying $abcn \neq 0$, although individual factors may switch from numerator to denominator or vice versa. We have seen this extendability in other clans as well, and it is typically associated with complicated wall-crossing phenomena. In this section, we discuss some of the extra Hurwitz-Belyi maps obtained by allowing negative parameters. As a special case of (4.2), one has the formula $\pi_{-a,-b,-c}(x) = \pi_{a,b,c}(x)^{-1}$. Using this symmetry, we can and will restrict attention to the half-space $c \geq 0$.

6.1. **Chambers.** Assuming none of the quantities in Table 5.1 vanish, the degree $N(a,b,c)$ of $\pi_{a,b,c}$ is the total of those quantities on the list $2(c-n)$, $2(a+c)$, $2(b+c)$, $2c$, $-a-2c$, $-b-2c$, and $a+b$ which are positive. This continuous, piecewise-linear function is homogeneous in the parameters a, b, and c, and so it can be understood by its restriction to $c=1$ via $N(a,b,c) = cN(a/c,b/c,1)$. The left half of Figure 6.1 is a contour plot of $N(\alpha,\beta,1)$. Thus, for $c \geq 4$ fixed, the minimum degree for $\pi_{a,b,c}$ is $4c$, occurring for all (a,b,c) with $(a/c,b/c)$ in the middle black triangle.

If $(a+c)(a+2c)(b+c)(b+2c)(n+c)(n+2c) = 0$, then there is a cancellation among at least a pair of factors, and $\pi_{a,b,c}$ has degree strictly less than $N(a,b,c)$. If $abcn = 0$ then, taking a limit, $\pi_{a,b,c}$ is still naturally defined, and again has degree strictly less than $N(a,b,c)$. Taking $c=1$, the lines given by the vanishing of the other nine linear factors in the discriminant formula are drawn in the right half of Figure 6.1. The complement of these lines has 31 connected components, called chambers. The middle chamber is the interior of the triangle given by $N(a,b,1)=4$. As indicated by the caption of Figure 6.1, one can also think of the right half of Figure 6.1 projectively. From this viewpoint, the line at infinity is given by the

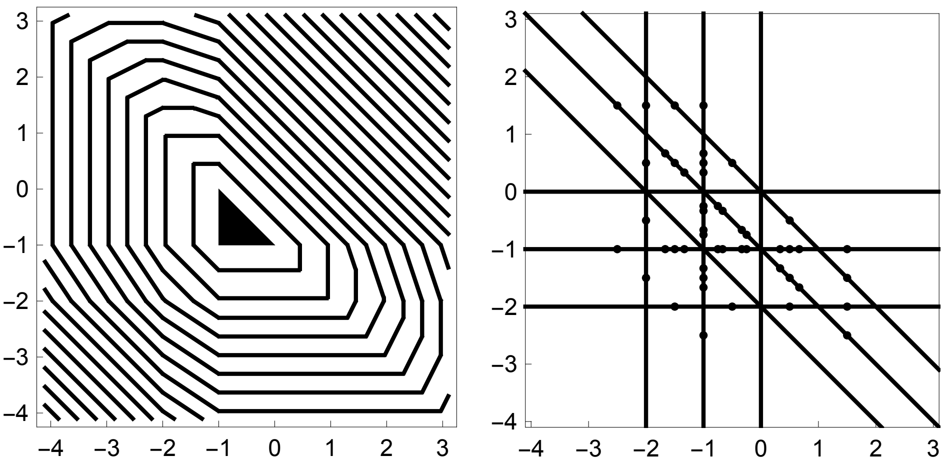


FIGURE 6.1. Left: the formal degree of $\pi_{\alpha,\beta,1}$, meaning the quantity degree$(\pi_{\alpha c,\beta c,c})/c$, drawn in the α-β plane. Contours range from 4 (middle triangle) to 24 (upper right corner). Right: The discriminant locus of the semicubical clan, drawn in the same (α, β) plane. Points are particular $(a/c, b/c)$-values giving covers appearing in Table 7.1.

vanishing of the remaining linear form, i.e., by $c = 0$. Our main reference [1] already illustrates some of this wall-crossing behavior in the context of distinct a, b, c, and d: the dessins with all parameters positive are described as being a *chardon*, while the dessins with parameters in certain other chambers are described as being a *pomme*.

6.2. **The central chamber and symmetric coordinates.** Each chamber corresponds to a different family of Hurwitz parameters, with corresponding rational functions $\pi_{a,b,c}$ being uniformly given by (3.4). To study the middle chamber, switch to new parameters $(u, v, w) = (c + a, c + b, c - n)$. In the new parameters, the quantity $c = u + v + w$ is still convenient, and we will use it regularly as an abbreviation. The middle chamber is given by the positivity of u, v, and w. We indicate the presence of the new parameters by capital letters, changing h to H, π to Π, and γ to Γ.

The normalized Hurwitz parameter (3.1) gets replaced by

(6.1)
$$H(u, v, w) = (S_{2c}, (2\,1^{2c-2}, c^2,\ 3_x\,1^{2c-3},\ (c-u)_0\,(c-v)_1(c-w)_\infty), (1, 1, 1, 1)).$$

Simply writing factors with the new parameters in different places corresponding to the new signs, (3.3) becomes

$$\Pi_{u,v,w}(x) = \frac{(-1)^{c-w}A^uB^vC^wD^c}{2^cc^{2c}(c-u)^{c-u}(c-v)^{c-v}(c-w)^{c-w}x^{c+u}(1-x)^{c+v}}. \tag{6.2}$$

The degree is $4c$ and the partition triple (3.2) changes to

$$\begin{aligned} \beta_0 &= u^2v^2w^2c^2, \\ \beta_1 &= 4^31^{4c-12}, \end{aligned} \tag{6.3}$$

$$\beta_\infty = (c+u)_0(c+v)_1(c+w)_\infty.$$

The symmetry (3.5) in the new parameters takes the similar form $\Pi_{u,v,w}(1-x) = \Pi_{v,u,w}(x)$. But now the symmetry

$$\Pi_{u,v,w}(1/x) = \Pi_{w,v,u}(x) \tag{6.4}$$

is equally visible.

In terms of a sheared version of Figure 6.1 in which the central triangle is equilateral, the symmetries just described generate the S_3 consisting of rotations and flips of this triangle. One has quadratic reduction as in (4.3) whenever two of the parameters are equal. One has cubic reduction as in (4.5) whenever one of the parameters is $2c$.

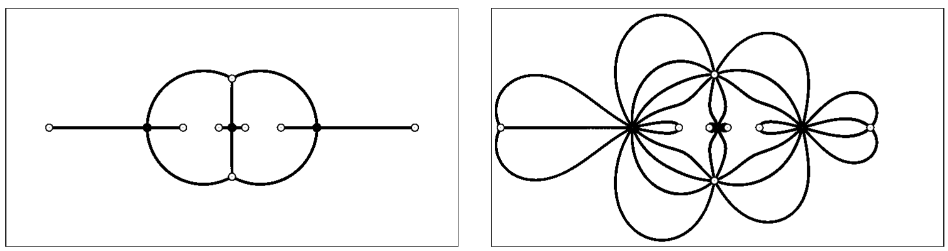


FIGURE 6.2. Left: $\Gamma(1,1,1)$. Right: $\Gamma(4,3,2)$.

For dessins, we take $\Gamma(u,v,w) = \Pi^{-1}_{u,v,w}([-\infty,0])$ with $[-\infty,0] =$ •—○ as before. Figure 6.2 is then direct analog of Figure 3.1. The black points on the real axis from left to right are, as before 0, ∞, 1. The unique white point above this axis is connected to 0, ∞, and 1 by respectively u, w, and v edges. To pass from $\Gamma(1,1,1)$ to $\Gamma(u,v,w)$, one replaces each edge by either u, v, or w parallel edges, illustrated by the example of $\Gamma(4,3,2)$.

6.3. **Degenerations.** While symmetric parameters are motivated by the central chamber, they are often better for analysis of the entire clan. As an example, we consider an aspect about degenerations over discriminantal lines, always excluding the intersections of these lines. To begin, consider the numerator of the logarithmic derivative $\Pi'_{u,v,w}(x)/\Pi_{u,v,w}(x)$. In conformity with $\beta_1 = 4^3 1^{m-12}$, one gets that this numerator is the cube of a cubic polynomial $\Delta(u,v,w,x)$. An easy computation gives $\Delta(u,v,w,x) =$

$$wx^3(w-c)(w+c)+3w(u-c)(w-c)x^2+3u(u-c)(w-c)x-u(u-c)(u+c). \tag{6.5}$$

Its discriminant has the completely symmetric form

$$\mathrm{disc}_x(\Delta(u,v,w,x)) = 2^2 3^3 uvw(u+c)^2(v+c)^2(w+c)^2c^3.$$

By symmetry, to understand the degenerations at the nine lines visible on the right half of Figure 6.1, one needs only to understand the degenerations at the horizontal lines. From bottom to top, the lines are given by $u=-c$, $u=0$, and $u=c$. Visibly from (6.5) the polynomials $\Delta(-c,v,w,x)$, $\Delta(0,v,w,x)$, and $\Delta(c,v,w,x)$ have 0 as a root of multiplicity 1, 2, and 3. Continuing this analysis, the Hurwitz-Belyi maps with parameter on these lines has β_1 of the form $4^2\,1^{m-8}$, $4\,2\,1^{m-6}$, and $3\,1^{m-1}$. The only discriminantal line not discussed yet is the line $c=0$ at infinity. Here again by symmetry, we need to consider only the case $(u, v, -m)$ with u and v positive satisfying $u + v = m$. In this case, one has $(\beta_0, \beta_1, \beta_\infty) = (u\,v, 2\,1^{m-2}, m)$.

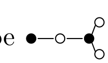
7. Moduli algebras

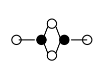
In Section 2 of [6], we saw some splitting of moduli algebras for two partition triples, with the unique factor of $\mathbb{Q}$ coming from a Hurwitz-Belyi map. The triples coming from the degeneration (4.4) and the semicubical clan itself (2.5) give us many moduli algebras which likewise have $\mathbb{Q}$ a factor. We computationally investigate small degree members of this collection here, as a further illustration that Hurwitz-Belyi maps are very special among all Belyi maps.

m	a	c	β_0	β_1	β_∞	μ	D
9	1	2	3321	$4\,2\,1^3$	54	$18+1$	$-\ \mathbf{2}^{37}\ \mathbf{3}^{26}\ \mathbf{5}^{12}\ 7^2$
12	1	3	4431	$4\,2\,1^6$	75	$39+1$	$\mathbf{2}^{84}\ \mathbf{3}^{41}\ \mathbf{5}^{30}\ \mathbf{7}^{26}\ 11^{10}$
15	1	4	5541	$4\,2\,1^9$	96	$60+1$	$\mathbf{2}^{105}\ \mathbf{3}^{107}\ \mathbf{5}^{55}\ 7^{17}\ 11^{10}$
15	2	3	5532	$4\,2\,1^9$	87	$60+1$	$-\ \mathbf{2}^{151}\ \mathbf{3}^{60}\ \mathbf{5}^{55}\ \mathbf{7}^{44}\ 13^{14}$

m	u	v	w	β_0	β_1	β_∞	μ	D
7	0	3	-1	322	$4\,2\,1$	511	$3+1$	$-\ \mathbf{2}^{3}\ \mathbf{3}\ \mathbf{5}^{2}\ 7$
9	0	2	1	3321	$4\,2\,1^3$	54	$18+1$	$-\ \mathbf{2}^{37}\ \mathbf{3}^{26}\ \mathbf{5}^{12}\ 7^2$
10	5	-4	0	5311	$4\,2\,1^4$	64	$28+1$	$\mathbf{2}^{56}\ \mathbf{3}^{41}\ \mathbf{5}^{20}\ 7^2$
10	0	4	-1	433	$4\,2\,1^4$	721	$28+1$	$\mathbf{2}^{50}\ \mathbf{3}^{34}\ 5^{14}\ \mathbf{7}^{19}$
10	5	-3	0	5221	$4\,2\,1^4$	73	$31+1$	$\mathbf{2}^{66}\ \mathbf{3}^{30}\ \mathbf{5}^{23}\ \mathbf{7}^{8}$
10	3	1	-2	33211	$4\,4\,1^2$	532	$33+1+1$	$\mathbf{2}^{69}\ \mathbf{3}^{42}\ \mathbf{5}^{23}\ 7^{21}$
11	0	5	-2	533	$4\,2\,1^5$	821	$38+1$	$\mathbf{2}^{113}\ \mathbf{3}^{45}\ \mathbf{5}^{28}\ 7^{5}\ 11^{10}$
12	0	3	1	4431	$4\,2\,1^6$	75	$39+1$	$\mathbf{2}^{84}\ \mathbf{3}^{41}\ \mathbf{5}^{30}\ \mathbf{7}^{26}\ 11^{10}$
12	5	-1	-2	552	$4\,4\,1^4$	72111	$41+1$	$\mathbf{2}^{74}\ \mathbf{3}^{18}\ \mathbf{5}^{31}\ \mathbf{7}^{31}\ 11^{15}$
12	6	-5	0	6411	$4\,2\,1^6$	75	$42+1$	$\mathbf{2}^{86}\ \mathbf{3}^{44}\ \mathbf{5}^{27}\ \mathbf{7}^{30}\ 11^{20}$

Table 7.1. Degrees μ and discriminants D of the moduli algebras coming from the partition triple (4.4) of $\pi^V_{a,c}$ and the partition triple of $\Pi_{u,v,w}$. The primes which are bad for the Hurwitz-Belyi map are in boldface.

Note for comparison that from (4.6), the dessin of $\pi^S_{1,1}$ has the shape  . The dessin of $\pi^S_{a,c}$ is obtained by replacing the middle edge by a parallel edges and the remaining three edges by c parallel edges. There are no other dessins that share the partition triple (4.6). So all moduli algebras of (4.6) are simply $\mathbb{Q}$. In fact the $\pi^S_{a,c}$ form a subclan of the four-parameter clan labeled C in [4]. Its dessins are obtained from Figure 13 of [4] by setting the indices (k, l, s, t) there equal to $(1, 1, c, a)$.

From (4.4), the dessin of $\pi^V_{1,1}$ is not a tree, being . The theory from [4] does not apply, and as a and c increase, more and more dessins share the partition triple (4.4). The four cases with $0 < a < c$ and $a + c \leq 5$ are in the top part of Table 7.1.

For our main case of $\pi_{a,b,c}$, the triple satisfying the conditions of Theorem 4.1 giving the lowest degree $m = 3(a+b+2c)$ is $(a,b,c) = (3,2,1)$ with $m = 21$. This case is beyond our computational reach. Allowing a and/or b to be negative, but staying off the discriminantal hyperplanes, the lowest degree is $m = 16$ from $(-1,-2,4)$. This case may be easier, but instead we allow (a,b,c) to be on a discriminantal hyperplane, excluding the extreme degenerations $abdn = 0$, as they give mass $\mu \leq 2$. We still require that the a, b, and c are distinct without a common factor. We switch to symmetric coordinates, so as to see the S_3 symmetries of the previous section clearly. Modulo these symmetries, Table 7.1 gives all cases with $m \leq 12$. The top line is the septic example from Sections 2 and 4 of [6] yet again.

The behavior summarized in Table 7.1 is similar to the behavior discussed in Section 2 of [6], but now the $(\beta_0, \beta_1, \beta_\infty)$ have been chosen to ensure splitting from the very beginning. In all cases but one, there is just one Belyi map defined over $\mathbb{Q}$ with the given partition triple, the Hurwitz-Belyi map. All the other Belyi maps are conjugate, the relevant Galois group being as large as possible, namely the symmetric group $S_{\mu-1}$. In the only case where there is extra splitting, the two rational Belyi maps are

$$\Pi_{3,1,-2}(x) = \frac{(8x-5)^2\left(8x^2-24x+15\right)^3\left(8x^2-8x+3\right)}{2^{14}(x-1)^3x^5(4x-3)^2},$$

$$\pi(x) = \frac{-(8x-5)^2\left(464x^2-840x+375\right)^3\left(2528x^2-4400x+1875\right)}{2^{16}5^5(x-1)^3x^5(4x-3)^2}.$$

Like $\Pi_{3,1,-2}(x)$, the unexplained rational factor $\pi(x)$ has bad prime set just $\{2,3,5\}$ and monodromy group S_{10}. An interesting problem of whether the moduli algebras $K_{u,v,w}$ just discussed can be treated in a uniform way with u, v and w appearing as parameters.

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