Chebyshev covers and exceptional number fields

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1. Context. A heuristic says that A_N or S_N number fields with discriminant $\pm 2^a 5^b$ and degree $N \geq 49$ "need a reason to exist."

2. Chebyshev covers. Define

$$T_{m,n}(x) = \frac{T_{m/2}(x)^n}{T_{n/2}(x)^m},$$

$$U_{m,n}(x) = \frac{U_{m/2}(x)^{2n}}{U_{n/2}(x)^{2m}}$$

in $\mathbb{Q}(x)$. These covers have *bad reduction at m and n only*, and *symmetric or alternating monodromy group*, a very unusual combination.

3. Exceptional Number fields, constructed by specializing Chebyshev covers. The fiber of $U_{125,128}(x)$ above 5 has discriminant of the form $\pm 2^a 5^b$ and Galois group S_{15875}.

1. Context. Two very important invariants of a degree N number field K are

- its associated *Galois group* $G \subseteq S_N$ and,

- its *discriminant* $D \in \mathbb{Z} - \{0\}$.

Example. Let $K = \mathbb{Q}[x]/(x^{13} - x - 1)$. Then

$G = S_{13}$
$D = 293959006143997 = 28201 \cdot 10423708597$

Random polynomials typically give G equal to all of S_N and D divisible by a large prime. However more sophisticated constructions can systematically give smaller G and/or tightly controlled D.

The *"refined" inverse Galois problem* is to identify the set $NF(G, D)$ of isomorphism classes of number fields with given invariants (G, D).

The refined inverse Galois problem is

- solved for abelian groups by cyclotomy
- solved in principle for solvable groups by class field theory
- solved in principle for groups like $GL_2(q)$ via modular forms
- approachable for groups like $GL_n(q)$, $Sp_{2n}(q)$, $O_n(q)$, $U_n(q)$ via motives and/or automorphic forms.

Ironically, the groups A_N and S_N are problematic cases, because it is hard to control D. The best general approach seems to be moduli fields of covers of the projective line.

For $N \geq 1$ and $v \in \{\infty, 2, 3, 5, 7, \ldots\}$ let $\lambda_{N,v}$ be the total mass of all degree N algebras over $\mathbb{Q}_v$.

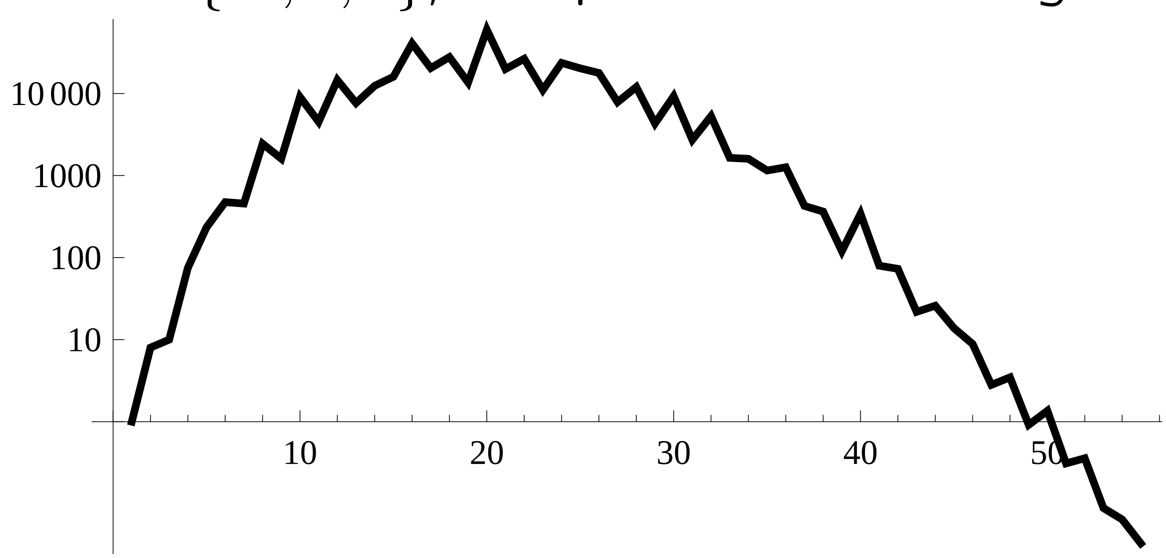
If $v = \infty$ the algebras are $\mathbb{R}^r\mathbb{C}^s$ with mass $\frac{1}{r!2^s s!}$. Summing over $r + 2s = N$ gives $\lambda_{N,\infty}$. The $\lambda_{N,\infty}$ decay superexponentially with N.

If $N < p$ then all ramification of $K/\mathbb{Q}_p$ is tame, and $\lambda_{N,p}$ is the number of partitions of N. In general, $\lambda_{N,p}$ grows roughly as $p^{N/(p-1)}$.

A heuristic says that given $S = \{\infty, p, \ldots, r\}$,

$$NF(\pm p^* \cdots r^*, A_N \text{ or } S_N) \approx \frac{1}{2}\lambda_{N,\infty}\lambda_{N,p}\cdots\lambda_{N,r}.$$

For $S = \{\infty, 2, 5\}$, the product on the right is



2A. Chebyshev covers. We work with indices $w \in \{1/2, 1, 3/2, 2, \ldots\}$. Define $T_w(x)$ and $U_w(x)$ in $\mathbb{Z}[\sqrt{x-2}, \sqrt{x+2}, x]$ by

$$\begin{aligned} T_w(z + z^{-1}) &= z^w + z^{-w}, \\ U_w(z + z^{-1}) &= z^w - z^{-w}. \end{aligned}$$

The right sides have $2w$ complex roots $e^{i\theta}$, equally spaced on the unit circle. Accordingly $T_w(x)$ and $U_w(x)$ each have w roots $2\cos\theta$, all in $[-2, 2]$ with roots at -2 and 2, if any, counted with multiplicity $1/2$.

As on the title slide, define

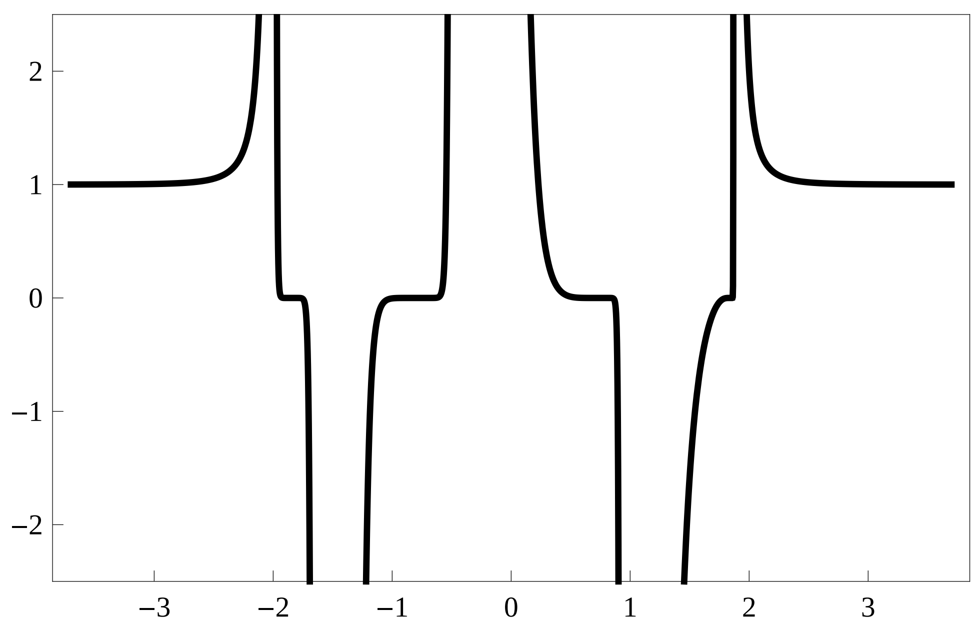
$$\begin{aligned} T_{m,n}(x) &= \frac{T_{m/2}(x)^n}{T_{n/2}(x)^m}, \\ U_{m,n}(x) &= \frac{U_{m/2}(x)^{2n}}{U_{n/2}(x)^{2m}}. \end{aligned}$$

Square roots cancel so that $T_{m,n}(x)$ and $U_{m,n}(x)$ are always in $\mathbb{Q}(x)$. WLOG, restrict to the case with $m < n$ relatively prime and, in the U case, not both odd.

As an example,

$$T_{8,9}(x) = \frac{(x^4 - 4x^2 + 2)^9}{(x+2)^4(x-1)^8(x^3 - 3x - 1)^8},$$

drawn in the window $|x| \le 3.7$, $|s| \le 2.5$:



The four zeros of multiplicity nine and the five poles of high even multiplicity are clearly visible.

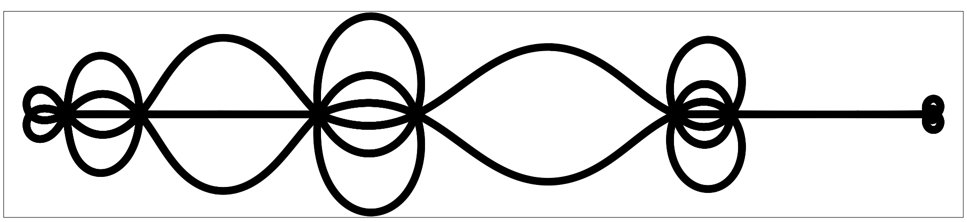
Both numerator and denominator have the form $x^{36} - 36x^{34} + 594x^{32} - 5952x^{30} + \cdots$, accounting for the rapidly-approached horizontal asymptote at $s = 1$.

Clearing denominators, we work with polynomials $T_{m,n}(s,x)$ and $U_{m,n}(s,x)$, e.g.

$$T_{m,n}(s,x) = T_{m/2}(x)^n - sT_{n/2}(x)^m$$

in the case of m, n of opposite parity.

In our example, as s increases from $-\infty$ to 0, the roots of $T_{8,9}(s,x)$ sweep out the following dessin, drawn in the region $[-2.1, 2.1] \times [-0.45, 0.45]$ of the complex x-plane:



The five poles $T_{8,9}(x)$ of are interspersed with the four zeros of $T_{8,9}(x)$. Edges connect poles to zeros in accordance with the diagram

$$\mathbf{4} \overset{4}{-} 9 \overset{5}{-} \mathbf{8} \overset{3}{-} 9 \overset{6}{-} \mathbf{8} \overset{2}{-} 9 \overset{7}{-} \mathbf{8} \overset{1}{-} 9 \overset{8}{-} \mathbf{8}.$$

The roots of $T_{8,9}(-1,x)$ mark the centers of the 36 edges while the roots of $T_{8,9}(1,x)$ mark the centers of the 28 bounded faces.

2B. Discriminants of Chebyshev covers.

Theorem. *One has discriminant formulas,*

$$\begin{aligned} \mathit{disc}_x(T_{m,n}(s,x)) &= \pm 2^* m^* n^* s^* (s-1)^* d_k^T(s), \\ \mathit{disc}_x(U_{m,n}(s,x)) &= \pm 2^* m^* n^* s^* (s-1)^* d_k^U(s), \end{aligned}$$

with the factor 2^* *missing if* m *and* n *are both odd. Here* $k = n - m$ *and the last factor has degree* $\lfloor (k-1)/2 \rfloor$.

We think of $T_{m,n}(x)$ and $U_{m,n}(x)$ as covering maps from the projective x-line to the projective s-line. The discriminant formula says their critical values are $s = 0$, 1, and ∞, and the roots of the relevant $d_k(s)$.

The first non-unital $d_k(s)$ are

$$\begin{aligned} d_3^T(s) &= s+1, & d_3^U(s) &= s+27, \\ d_4^T(s) &= s+4, & d_4^U(s) &= s-16. \end{aligned}$$

2C. QuasiChebyshev covers. For $k \leq 2$, the covers $T_{m,n}$ and $U_{m,n}$ are three-point covers, i.e. Belyi maps. However they are *not* determined by their triples of ramification partitions.

For example, the partitions for $T_{8,9}$ are

$$\begin{aligned} \lambda_0 &= (9,9,9,9), \\ \lambda_1 &= (8,1,\cdots,1), \\ \lambda_\infty &= (8,8,8,8,4). \end{aligned}$$

In general, quasiChebyshev covers are indexed by planar trees marked by *polar multiplicities only.* For the above partitions, two trees have extra automorphisms:

$$\mathbf{8}-\mathbf{8}-\mathbf{4}-\mathbf{8}-\mathbf{8} \qquad\qquad \begin{array}{ccc} & \mathbf{8} & \\ & | & \\ \mathbf{8}- & \mathbf{4} & -\mathbf{8} \\ & | & \\ & \mathbf{8} & \end{array}$$

Eight have only the identity automorphism:

$$4-8-8-8-8$$

(Chebyshev)

$$\begin{array}{ccccc} & & & & 8\\ & & & & \vert\\ 4 & - & 8 & - & 8\\ & & & & \vert\\ & & & & 8 \end{array}$$

$$8-4-8-8-8$$

$$\begin{array}{ccccc} & & & & 8\\ & & & & \vert\\ 8 & - & 4 & - & 8\\ & & & & \vert\\ & & & & 8 \end{array}$$

$$\begin{array}{ccccc} & & 8 & & \\ & & \vert & & \\ 4 & - & 8 & - & 8\\ & & \vert & & \\ & & 8 & & \end{array}$$

$$\begin{array}{ccccc} 8 & & & & \\ \vert & & & & \\ 4 & - & 8 & - & 8\\ \vert & & & & \\ 8 & & & & \end{array}$$

$$\begin{array}{ccccc} & & 8 & - & 8\\ & & \vert & & \\ 4 & - & 8 & & \\ & & \vert & & \\ & & 8 & & \end{array}$$

$$\begin{array}{ccccc} & & 8 & & \\ & & \vert & & \\ 4 & - & 8 & & \\ & & \vert & & \\ & & 8 & - & 8 \end{array}$$

To get the quasiChebyshev covers indexed by the last eight trees, require that $F^{-1}(1)$ contain ∞ with multiplicity eight, where

$$F(x) = \frac{(x^4 + a_1x^2 + a_2x + a_3)^9}{(x+2)^4(x^4 + c_1x^3 + c_2x^2 + c_3x + c_4)^8}.$$

The resulting system of seven equations in seven unknowns reduces to

$$\begin{array}{r}(a_1 + 4)(35a_1^7 + 2380a_1^6 + 38192a_1^5 + \\ 236480a_1^4 + 928000a_1^3 + 3015680a_1^2 \\ -3993600a_1 - 16564224) = 0.\end{array}$$

The first factor corresponds to $T_{8,9}(x)$. The second factor has field discriminant

$$-2^4 3^5 5^6 7^2 11^5 19^3.$$

All computed cases are exactly like this: the general theory of *dessins d'enfants* gives no reason for the the Chebyshev covers to split off; it gives no reason for the them to be ramified only at the primes dividing mn.

2D. Monodromy.

Theorem. *If $F : \mathbf{P}^1_x \to \mathbf{P}^1_s$ is a degree N quasi-Chebyshev cover without non-identity automorphisms, then its monodromy group is A_N or S_N.*

Sketch of proof. Since the top curve is connected, the monodromy group G is irreducible. The ramification partitions λ_0, λ_∞ allow there to be a factorization $\mathbf{P}^1_x \stackrel{N_1}{\to} \mathbf{P}^1 \stackrel{N_2}{\to} \mathbf{P}^1_s$ only if the first map has just two ramification points. Thus if there are no non-trivial automorphisms, the monodromy group has to be primitive. By the classification of primitive Galois groups, the only ones containing an element of cycle type $\lambda_1 = (m, 1, 1, \ldots, 1, 1)$ are A_N or S_N. $\square$

As a consequence, $T_{m,n}(s, x)$ and $U_{m,n}(s, x)$ have Galois group A_N or S_N for almost all $s \in \mathbb{Q}$. However, in the most interesting cases, e.g. $s = \pm 1$, the Galois group is still in doubt.

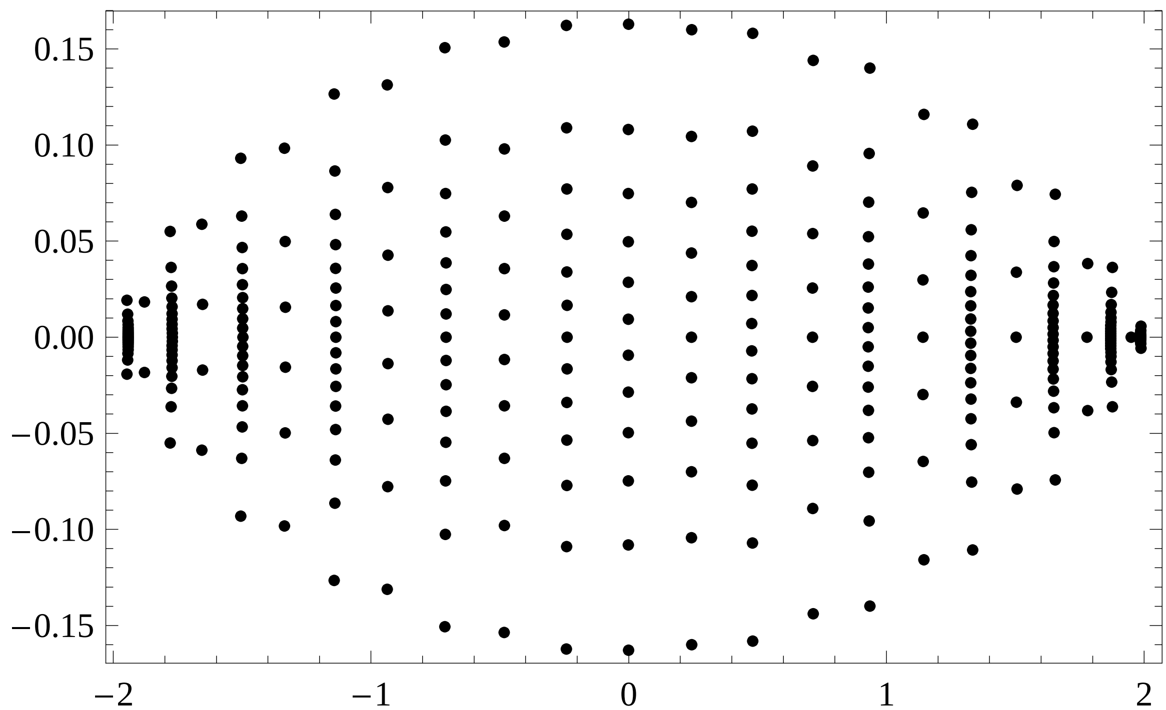
3. Exceptional Fields. For each S, the heuristic gives a cutoff past which A_N and S_N "need a reason to exist." We call fields in this range *exceptional.*

For $S = \{\infty, 2, 3\}$, the exceptional range is $[62, \infty)$.

Specializing $T_{8,9}(s, x)$ gives an S_{28} field (at $s = 1$), an S_{35} field (at $s = 2$), four A_{36} fields and fifteen S_{36} fields. Specializing $U_{8,9}(s, x)$ gives four A_{64} fields, and seventeen S_{64} fields. Only the fields belonging to the second collection are exceptional according to our formal definition.

However, in general, field discriminants of specializations are both low and regularly behaved. For example, the S_{28} field has discriminant $2^{83}3^{54}$, while the largest discriminant allowed by local bounds is $2^{118}3^{107}$. Thus even the fields we classify as non-exceptional are quite remarkable.

For $S = \{\infty, 3, 5\}$, the exceptional range is $[38, \infty)$. The polynomial $T_{25,27}(1, x)$ has degree 300 and roots as follows.



However, the Galois group is not A_{300} or S_{300} because there is a triangular structure to the roots. Modding out by rotating the triangle gives a polynomial $T^{\text{red}}_{25,27}(1, x) = x^{100} - 625x^{99} + \cdots$. Using that this polynomial is irreducible, has square discriminant and factors modulo 2 as $\lambda_2 = (71, 14, 12, 3)$, we get that it has Galois group A_{100}.

For $S = \{\infty, 2, 5\}$, the exceptional range is $[49, \infty)$. The polynomial $U_{125,128}(5, x)$ has degree 15875. Both ABC triples involved in its construction involve 3:

$$\begin{aligned} 5^3 + 3 &= 2^7, \\ 3^3 + 5 &= 2^5. \end{aligned}$$

Despite this, even the polynomial discriminant doesn't involve 3 as it is $-2^{130729}5^{63437}$. The first four factor partitions are

$$\begin{aligned} \lambda_3 &= 10194, 3365, 2123, 155, 20, 10, 5, 3 \\ \lambda_7 &= 7332, 2492, 1642, 1388, 1077, 1011, 818, 72, 24, 10, 9 \\ \lambda_{11} &= 9784, 3238, 1272, 648, 480, 143, 139, 133, 17, 12, 9 \\ \lambda_{13} &= 6808, 4493, 3803, 626, 74, 39, 13, 8, 6, 3, 2 \end{aligned}$$

A 1918 criterion of Manning says that a degree N primitive group containing an element of cycle type $P^q 1^k$ for P prime with $P \geq 2q - 1$ and $k \geq 4q - 1$ is A_N or S_N. The data suffices to prove that the Galois group of $U_{125,128}(5, x)$ is S_{15875}.

Main References

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