DISCRIMINANTS OF SOME PAINLEVÉ POLYNOMIALS

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Abstract. We derive explicit formulas for the discriminants of the Yablonsky-Vorobiev polynomials $P_m(x)$, the biHermite polynomials $H_{m,n}(x)$ and the Okamoto polynomials $Q_{m,n}(x)$, as well as some related resultants. In all three cases, the discriminant and related resultants factor into a product of small primes only. Our formula in the biHermite case reduces when $n = 1$ to the nineteenth-century formula for the discriminant of a Hermite polynomial. Our other two discriminant formulas do not have direct antecedents.

1. Introduction

The Painlevé hierarchy was discovered by Painlevé and Gambier around 1900. It consists of six second-order non-linear ordinary differential equations I-VI, each depending on appropriate parameters. In brief, one can think of Painlevé VI as a non-linear analog of the usual linear hypergeometric equation. The others are limiting degenerations of Painlevé VI, thus analogs of the confluent hypergeometric equation:

$$\begin{tikzcd}
 & & \mathrm{III} \arrow[dl] & & \\
\mathrm{I} & \mathrm{II} \arrow[l] & & \mathrm{V} \arrow[ul] \arrow[dl] & \mathrm{VI}. \arrow[l] \\
 & & \mathrm{IV} \arrow[ul] & &
\end{tikzcd}$$

As one moves rightward in this diagram, one adds a parameter at each step, so that Painlevé I involves 0 parameters and Painlevé VI involves 4.

Most of the large and rapidly-expanding literature on the Painlevé equations is connected with mathematical physics. However, we are looking here at number-theoretic phenomena and familiarity with the literature is not necessary to understand this paper. For readers who nonetheless would like some general background, we recommend [IKSY], [Co], and [Um] as places to start. The textbook [IKSY] characterizes the six Painlevé equations as "the most important non-linear ordinary differential equations." The instructional conference proceedings [Co] supports this characterization by describing many situations in mathematical physics which ultimately reduce to a Painlevé equation. The article [Um] is an easy-to-read introduction aimed at pure mathematicians.

The bulk of this paper concerns Painlevé II and IV:

$$y'' = 2y^3 + xy + a \tag{1.1}$$

$$y'' = \frac{(y')^2}{2y} + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - a)y + \frac{b}{y}. \tag{1.2}$$

Around 1960, Yablonsky [Ya] and Vorobiev [Vo] found rational solutions to Painleve II. For m an integer, they defined polynomials $P_m \in \mathbf{Z}[x]$ and

$$f_m = \frac{d}{dx} \log \left(\frac{P_{m+1}}{P_m} \right).$$

Then $s_P f_m(s_P x)$ with $s_P = 2^{-2/3}$ is a solution to Painlevé II with $a = -m - 1$.

In the 1980's, Murata [Mu] found a family of solutions to Painlevé IV, indexed by $(m,n) \in \mathbf{Z}^2_{\geq 0}$. He defined polynomials $H_{m,n}$ and

$$h_{m,n} = \frac{d}{dx} \log \left(\frac{H_{m,n+1}}{H_{m,n}} \right) = \frac{H_{m+1,n} H_{m-1,n+1}}{H_{m,n+1} H_{m,n}}.$$

The function $s_H h_{m,n}(s_H x)$ with $s_H = \sqrt{-2}$ solves Painlevé IV for $a = m + 2n + 1$ and $b = -2m^2$. We call the polynomials $H_{m,n}$ biHermite polynomials because they depend on the two indices m and n and $H_{m,1}(x) = i^{-m} H_{1,m}(ix)$ both coincide with the classical Hermite polynomial $H_m(x)$.

Also in the 1980's, Murata [Mu] found another family of solutions to Painlevé IV, this one indexed by $(m,n) \in \mathbf{Z}^2$. This family forms a generalization of two sequences of polynomials found several years earlier by Okamoto. Murata defined polynomials $Q_{m,n} \in \mathbf{Z}[x]$ and

$$g_{m,n} = \frac{d}{dx} \log \left(\frac{Q_{m,n+1}}{Q_{m,n}} \right) + x = \frac{Q_{m+1,n} Q_{m-1,n+1}}{Q_{m,n+1} Q_{m,n}}.$$

Then $s_Q g_{m,n}(s_Q x)$ with $s_Q = \sqrt{-2/3}$ solves Painlevé IV for $a = m + 2n$ and $b = (-2/9)(3m-1)^2$.

It should be noted that notations and normalizations vary somewhat in the literature. We have opted to keep the most standard form of the Painlevé equations (1.1) and (1.2). Our versions of P_m, $H_{m,n}$, and $Q_{m,n}$ are monic and well-behaved at the primes 2 and 3. These good aspects of our normalizations force the presence of the unattractive irrational scale factors s_P, s_H, and s_Q. However, because of the symmetries (3.2), (4.3), (5.3), all rational solutions are in fact in $\mathbf{Q}(x)$.

In §2, we review the formalism of discriminants and resultants. Sections §3, §4, and §5 present our discriminant and resultant formulas for P_m, $H_{m,n}$, and $Q_{m,n}$ respectively. As our title suggests, our primary interest is discriminants. However our inductive method of proof requires us to compute related resultants as well. The induction proceeds by expressing a given discriminant or resultant as a product of earlier discriminants, earlier resultants, and small integers. Accordingly, all the final formulas have a common structure: discriminants and resultants are expressed as products of small integers to large powers. In carrying out the induction, we make fundamental use of quadratic contiguity relations among the polynomials in question.

The polynomials we consider here exhibit great regularity beyond what we capture in our discriminant and resultant formulas. Moreover, the polynomials P_m and $H_{m,n}$ are limiting cases of polynomials related to solutions to Painlevé III, V, and VI. In §6 we briefly indicate several ways in which our formulas here are fitting into an emerging larger picture.

We have implemented P_m, $H_{m,n}$ and $Q_{m,n}$ in a small *Mathematica* file. This file can be used as an aid in reading this paper. It is available at `http://cda.mrs.umn.edu/~roberts`.

2. Discriminants and resultants

Let $f(x) = a_0x^M + \cdots$ be a degree M polynomial in $\mathbf{Z}[x]$ with complex roots $\alpha_1, \ldots, \alpha_M$. Then the discriminant of f is defined to be

$$Disc(f) \quad = \quad a_0^{2M-1} \prod_{1\leq i<j\leq M} (\alpha_i - \alpha_j)^2 \in \mathbf{Z}. \tag{2.1}$$

Let $g(x) = b_0x^N + \cdots$ be a degree N polynomial in $\mathbf{Z}[x]$ with complex roots $\beta_1, \ldots, \beta_N$. Then the resultant of f and g is

$$Res(f,g) = a_M^N b_N^M \prod_{i=1}^{M}\prod_{j=1}^{N}(\alpha_i - \beta_j) \in \mathbf{Z}. \tag{2.2}$$

The connection between resultants and discriminants is

$$Disc(f) = (-1)^{M(M-1)/2} Res(f, f'). \tag{2.3}$$

While discriminants are just a special case of resultants, they are worthy of special attention.

It is clear from (2.2) that resultants behave homogeneously on constant polynomials, generalize evaluation, are multiplicative, and are symmetric up to sign:

$$Res(f,c) = c^{\deg(f)} \quad (2.4)$$

$$Res(f,x-c) = (-1)^{\deg(f)} f(c) \quad (2.5)$$

$$Res(f,g_1g_2) = Res(f,g_1)Res(f,g_2) \quad (2.6)$$

$$Res(f,g) = (-1)^{\deg(f)\deg(g)} Res(g,f). \quad (2.7)$$

Another basic fact about resultants, not so immediate from (2.2), is that

$$Res(f,qf+r) = Res(f,r). \quad (2.8)$$

This fact allows one to compute resultants by the Euclidean algorithm. The initial step in all three of our proofs is several applications of (2.8). In these applications qf represents one or two terms in a quadratic relation. So these terms are not seen in our discriminant and resultant calculations.

The relations (2.3)-(2.8) completely characterize the functionals $Disc(\cdot)$ and $Res(\cdot,\cdot)$. We will not make further reference to (2.1) and (2.2) until §6. Thus we will be working purely over $\mathbf{Z}$, without any reference to complex roots.

Denote the absolute version of discriminants and resultants as follows

$$\begin{aligned} D(f) &= |Disc(f)| \\ R(f,g) &= |Res(f,g)|. \end{aligned}$$

In the absolute setting, the formalism of discriminants and resultants simplifies as the signs in (2.3), (2.5) and (2.7) disappear. To make our theorems and proofs more readable we will present them in the absolute setting; thus we will ignore minus signs and work with positive numbers only. Of course, it is easier to do the reverse, that is to ignore magnitudes and work with signs only. Doing this yields

$$\begin{array}{rcll} Disc(P_m) < 0 & \Leftrightarrow & m \equiv 2\ (4) & \\ Disc(H_{m,n}) < 0 & \Leftrightarrow & (m,n) \equiv & (1,2),(1,3),(3,2),(3,3) \quad (4) \\ Disc(Q_{m,n}) < 0 & \Leftrightarrow & (m,n) \equiv & \begin{array}{l}(2,0),(2,1),(2,2),(2,3),\\(1,2),(1,3),(3,0),(3,1)\end{array} \quad (4). \end{array}$$

The signs of the resultants we treat are also determined by their indices modulo 4.

3. Yablonsky-Vorobiev polynomials

For $m \in \mathbf{Z}$, the Yablonsky-Vorobiev polynomials are defined by

$$\begin{aligned} P_0 &= 1 \\ P_1 &= x \\ P_{m+1}P_{m-1} &= P'_m P'_m - P''_m P_m + x P_m P_m. \end{aligned} \quad (3.1)$$

From this definition it is apparent that the P_m are rational functions. It is not apparent but it is true that they are polynomials. The analogous remarks hold also in the biHermite and Okamoto cases.

The polynomial P_m has degree

$$\Delta_m = \frac{m(m+1)}{2}$$

and satisfies the symmetry

$$P_m(\omega x) = \omega^{\Delta_m} P_m(x) \tag{3.2}$$

for $\omega \in \mathbf{C}$ a third root of unity.

Besides the quadratic relation (3.1), we need the following quadratic relations.

$$\begin{aligned}
P_{m+2}P_{m-2} &= (4-(2m+1)^2)P'_m P'_m + \\
&\quad (x^4+(2m+1)^2 x)P_m P_m - \\
&\quad 4x^2 P_m P'_m + (2m+1)^2 P_m P''_m && (3.3)\\
P_{m+2}P_{m-1} &= -(2m+3)P'_m P_{m+1} + \\
&\quad (2m+1)P'_{m+1}P_m + x^2 P_m P_{m+1}. && (3.4)
\end{aligned}$$

These can be derived from the determinant representation [KYO, Prop. 4.1] of P_m, following the model provided by the proof of [KYO, Prop. 2.2].

One has the symmetry $P_m = P_{1-m}$ so we will henceforth restrict attention to $m \in \mathbf{Z}_{\geq 0}$.

Theorem 3.1. *One has the discriminant formula*

$$D(P_m) = \prod_{j=3,5,\ldots}^{2m-1} j^{j(2m+1-j)^2/4}$$

and the resultant formulas

$$\begin{aligned}
R(P_m, P_{m-1}) &= \prod_{j=3,5,\ldots}^{2m-3} j^{j(2m+1-j)(2m-1-j)/4} \\
R(P_m, P_{m-2}) &= (2m-1)^{\Delta_{m-1}} \prod_{j=3,5,\ldots}^{2m-5} j^{j(2m+1-j)(2m-3-j)/4}.
\end{aligned}$$

Here the products run over odd integers, as indicated.

Proof. Applying $R(P_m,\cdot)$ to (3.1), (3.3), and (3.4) yields

$$R(P_{m+1},P_m)R(P_m,P_{m-1}) = D(P_m)^2 \tag{3.5}$$

$$R(P_{m+2},P_m)R(P_m,P_{m-2}) = ((2m-1)(2m+3))^{\Delta_m} D(P_m)^2 \tag{3.6}$$

$$R(P_{m+2},P_m)R(P_m,P_{m-1}) = (2m+3)^{\Delta_m} R(P_{m+1},P_m)D(P_m). \tag{3.7}$$

As abbreviations, define

$$\begin{aligned} d_m &= D(P_m) \\ r_m &= R(P_m, P_{m+1}) \\ (2m+3)^{\Delta_{m+2}} \rho_m &= R(P_m, P_{m+2}), \end{aligned}$$

so that

$$(2m-1)^{\Delta_m} \rho_{m-2} = R(P_{m-2}, P_m).$$

These abbreviations absorb most of the factors involving $(2m+3)$ and $(2m-1)$, and (3.5), (3.6), (3.7) can be written respectively as

$$d_m^{-2} r_m = r_{m-1}^{-1} \tag{3.8}$$

$$d_m^{-2} \rho_m = \rho_{m-2}^{-1} (2m+3)^{-2m-3} \tag{3.9}$$

$$d_m^{-1} r_m^{-1} \rho_m = r_{m-1}^{-1} (2m+3)^{-2m-3}. \tag{3.10}$$

The exponents of the three new variables in the three equations just displayed are as indicated in the following 3-by-3 matrix

	d_m	r_m	ρ_m
(3.8)	-2	1	0
(3.9)	-2	0	1
(3.10)	-1	-1	1

The inverse of this 3-by-3 matrix is

	(3.8)	(3.9)	(3.10)
d_m	-1	1	-1
r_m	-1	2	-2
ρ_m	-2	3	-2

Accordingly, to isolate the new variable d_m we form the multiplicative combination $(3.8)^{-1}(3.9)^1(3.10)^{-1}$ and similarly for r_m and ρ_m. The result is

$$d_m = F_m^1 r_{m-1} \tag{3.11}$$

$$r_m = F_m^2 r_{m-1} \tag{3.12}$$

$$\rho_m = F_m^3 r_{m-1} (2m+3)^{-2m-3}, \tag{3.13}$$

where we have defined $F_m = r_{m-1}/\rho_{m-2}$. Using (3.12) twice and (3.13) once we deduce a two-step recursion:

$$\begin{aligned} F_m &= \frac{r_{m-1}}{\rho_{m-2}} \\ &= \frac{F_{m-1}^2 r_{m-2}(2m-1)^{2m-1}}{F_{m-2}^3 r_{m-3}} \\ &= \frac{F_{m-1}^2 F_{m-2}^2 r_{m-3}(2m-1)^{2m-1}}{F_{m-2}^3 r_{m-3}} \\ &= \frac{F_{m-1}^2 (2m-1)^{2m-1}}{F_{m-2}}. \end{aligned}$$

By direct calculation one has $F_2 = 3^3$ and $F_3 = 3^6 5^5$. The unique solution of the two-step recursion satisfying these initial conditions is

$$F_m = \prod_{j=3,5,\dots}^{2n-1} j^{j(2m+1-j)}. \tag{3.14}$$

Combining (3.14) with (3.11)-(3.13) and elementary steps gives the formulas in Theorem 3.1. □

4. BiHermite polynomials

Our basic reference for biHermite polynomials is [NY], especially Theorem 4.2 which gives the quadratic relations we need. However we will renormalize the polynomials $H_{m,n}^{NY}$ there via

$$H_{m,n}(x) = \frac{H_{m,n}^{NY}(x/\sqrt{3})}{\sqrt{3}^{mn+m(m-1)+n(n-1)} \prod_{j=1}^{m-1} j^{m-j} \prod_{j=1}^{n-1} j^{n-j}}.$$

The polynomials $H_{m,n}$ are defined for $m, n \in \mathbf{Z}_{\geq 0}$ and characterized by

$$\begin{aligned} H_{m,0} &= H_{0,n} = 1 \\ H_{1,1} &= x \\ nH_{m,n+1}H_{m+1,n-1} &= H_{m,n}H'_{m+1,n} - H_{m+1,n}H'_{m,n} \qquad (4.1) \\ mH_{m+1,n}H_{m-1,n+1} &= H_{m,n}H'_{m,n+1} - H_{m,n+1}H'_{m,n} \qquad (4.2) \end{aligned}$$

One has the degree formula

$$\text{degree}(H_{m,n}) = mn$$

and the symmetry

$$H_{m,n}(ix) = i^{mn} H_{n,m}(x). \tag{4.3}$$

Finally Theorem 4.2 of [NY] gives not only the quadratic relations (4.1) and (4.2), but also

$$\begin{aligned} (4.4)\qquad H'_{m,n}H'_{m,n} - H_{m,n}H''_{m,n} &= -m(H_{m+1,n}H_{m-1,n} - H^2_{m,n}) \\ (4.5)\qquad H'_{m,n}H'_{m,n} - H_{m,n}H''_{m,n} &= n(H_{m,n+1}H_{m,n-1} - H^2_{m,n}). \end{aligned}$$

Define

$$e(j) = \begin{cases} j^2 - 2(m-j)(n-j) & \text{if } j \le \min(m,n) \\ \min(m,n)^2 & \text{if } \min(m,n) \le j \le \max(m,n) \\ (m+n-j)^2 & \text{if } \max(m,n) \le j. \end{cases}$$

For $n \in \mathbf{Z}_{\ge 0}$ define also

$$\begin{aligned} n^* &= \prod_{j=1}^{n} j^j \\ n^{**} &= \prod_{m=1}^{n} m^*. \end{aligned}$$

We adopt the convention that $(-1)^* = 0^* = (-1)^{**} = 0^{**} = 1$.

Theorem 4.1. *The discriminant of $H_{m,n}$ is*

$$D(H_{m,n}) = \prod_{j=2}^{m+n-1} j^{je(j)}.$$

Related resultants are

$$\begin{aligned} R(H_{m,n}, H_{m+1,n}) &= \frac{(m+n-1)^{**}}{(m-1)^{**}m^{*n}(n-1)^{**}} D(H_{m,n}) \\ R(H_{m,n}, H_{m,n+1}) &= \frac{(m+n-1)^{**}}{(m-1)^{**}n^{*m}(n-1)^{**}} D(H_{m,n}) \\ R(H_{m,n}, H_{m+1,n-1}) &= \frac{(n-1)^{*m}}{m^{*n}} D(H_{m,n}). \end{aligned}$$

Proof. The theorem is true whenever $mn = 0$, as both sides of all four equations are 1. We totally order the remaining index set $\mathbf{Z}^2_{\ge 1}$ by saying

$$(\mu,\nu) < (m,n) \Leftrightarrow \left(\begin{array}{ccccccc} \mu+\nu & < & m+n & & & & \\ & \text{or} & & & & & \\ \mu+\nu & = & m+n & \text{and} & \mu & < & m \end{array} \right).$$

The proof proceeds inductively with respect to this order. So suppose the theorem is proved up through, but not including, the index $(m,n) \in \mathbf{Z}^2_{\ge 1}$.

Abbreviate

$$\begin{aligned} a_{m,n} &= R(H_{m,n}, H_{m+1,n}) \\ b_{m,n} &= R(H_{m,n}, H_{m,n+1}) \\ c_{m,n} &= R(H_{m,n}, H_{m+1,n-1}) \\ d_{m,n} &= D(H_{m,n}). \end{aligned}$$

Abbreviate further

$$\begin{array}{rclcrcl} a &=& a_{m,n} & \quad & A &=& a_{m-1,n} \\ b &=& b_{m,n} & & B &=& b_{m,n-1} \\ c &=& c_{m,n} & & C &=& c_{m-1,n+1} \\ d &=& d_{m,n}. & & & & \end{array}$$

The three index-pairs in the right column are all $< (m,n)$. So we can regard A, B, C as old variables and a, b, c, d as new variables.

Applying $R(H_{m,n}, \cdot)$ to (4.1), (4.2), (4.4), and (4.5), one gets

$$\begin{aligned} a^{-1}bcd^{-1} &= n^{-mn} \\ ab^{-1}d^{-1} &= C^{-1}m^{-mn} \\ a^{-1}d^2 &= Am^{mn} \\ b^{-1}d^2 &= Bn^{mn}. \end{aligned}$$

We follow the basic pattern of (3.8)-(3.13). The exponents on the left side this time form a 4-by-4 matrix; this matrix is again invertible over the integers. Inverting this matrix, and taking corresponding multiplicative combinations, we get

$$\begin{aligned} a &= A^{-3}B^2C^2n^{2mn}m^{-mn} \\ b &= A^{-2}BC^2n^{mn} \\ c &= A^{-2}B^2Cn^{mn}m^{-mn} \\ d &= A^{-1}BCn^{mn}. \end{aligned}$$

Plugging in the known formulas on the right and carrying out a number of elementary steps yields the formulas to be proved for the quantities on the left. □

5. Okamoto polynomials

Our basic reference for Okamoto polynomials is again [NY], especially Theorem 4.2 which again give the quadratic relations we need. However we will reindex the polynomials $Q_{m,n}^{NY}$ there by

$$Q_{m,n,r} = Q_{m,n} = Q_{m+n,n}^{NY}.$$

Here r is a redundant parameter always satisfying

$$m+n+r=1.$$

We include r sometimes to see an S_3 symmetry more clearly. For example,

$$d_{m,n} = \frac{1}{2}(m^2+n^2+r^2-1) \tag{5.1}$$

is the degree of $Q_{m,n}$.

The polynomials $Q_{m,n} \in \mathbf{Z}[x]$ are defined for $m, n \in \mathbf{Z}$ and characterized by the following equations:

$$\begin{aligned}
Q_{0,0} &= Q_{1,0} = Q_{0,1} = 1 \\
Q_{1,1} &= x \\
Q_{m+1,n}Q_{m-1,n} &= (x^2+2m+n-1)Q_{m,n}^2 + \\
&\quad Q''_{m,n}Q_{m,n} - Q'_{m,n}Q'_{m,n} \qquad (5.2)\\
Q_{m,n}(ix) &= i^{d_{m,n}}Q_{n,m}(x). \qquad (5.3)
\end{aligned}$$

One has the further symmetry

$$Q_{m,n,r} = Q_{r,m,n} = Q_{n,r,m}. \tag{5.4}$$

This symmetry means that every Okamoto polynomial can be written $Q_{m,n}$ with $mn \geq 0$.

From Theorem 4.2 of [NY] one gets not only the quadratic relation (5.2), but also

$$Q'_{m,n}Q_{m+1,n} = Q_{m,n}Q'_{m+1,n} - xQ_{m,n}Q_{m+1,n} + Q_{m+1,n-1}Q_{m,n+1} \tag{5.5}$$

$$\begin{aligned}
0 &= (3r-1)Q_{m+1,n-1,r}Q_{m-1,n+1,r} + \\
&\quad (3m-1)Q_{m,n+1,r-1}Q_{m,n-1,r+1} + \\
&\quad (3n-1)Q_{m-1,n,r+1}Q_{m+1,n,r-1}.
\end{aligned} \tag{5.6}$$

For $x, x' \in \mathbf{Z}$ with $|x-x'| \leq 1$ define

$$f_{x,x'} = \begin{cases} \displaystyle\prod_{j=2,5,\ldots}^{3\min(x,x')-4} j^{j(x-(j+1)/3)(x'-(j+1)/3)} & \text{if } x, x' \geq 2 \\ \displaystyle\prod_{j=4,7,\ldots}^{3\min(|x|,|x'|)-2} j^{j(|x|-(j-1)/3)(|x'|-(j-1)/3)} & \text{if } x, x' \leq -2 \\ 1 & \text{else.} \end{cases}$$

Clearly $f_{x,x'} = f_{x,x'}$. Also, straightforward calculation yields

$$f_{x-1,x} f_{x+1,x} = f_{x,x}^2 \tag{5.7}$$

$$f_{x,x-1}^2 f_{x+1,x+1} = f_{x,x+1}^2 f_{x-1,x-1} (3x-1)^{3x-1}. \tag{5.8}$$

Theorem 5.1. *The discriminant of $Q_{m,n,r}$ depends separately on m, n, and r via*

$$D(Q_{m,n,r}) = f_{m,m} f_{n,n} f_{r,r}. \tag{5.9}$$

If $\{m - m', n - n', r - r'\} = \{-1, 0, 1\}$ then the corresponding resultant factors as

$$R(Q_{m,n,r}, Q_{m',n',r'}) = f_{m,m'} f_{n,n'} f_{r,r'}. \tag{5.10}$$

Proof. A difference between the present Okamoto case and the biHermite case of §4 is that here we do not have a fact analogous to $H_{m,0} = H_{0,n} = 1$. Accordingly, we have to replace the induction there by a different inductive argument here.

Abbreviate

$$\begin{aligned} D_{m,n,r} &= D(Q_{m,n,r}) \\ R_{m,n,r}^{m',n',r'} &= R(Q_{m,n,r}, Q_{m',n',r'}). \end{aligned}$$

Applying $R(Q_{m,n,r}, \cdot)$ to (5.2) and (5.5) respectively gives

$$D_{m,n,r}^2 = R_{m,n,r}^{m+1,n,r-1} R_{m,n,r}^{m-1,n,r+1} \tag{5.11}$$

$$D_{m,n,r} R_{m,n,r}^{m+1,n,r-1} = R_{m,n,r}^{m+1,n-1,r} R_{m,n,r}^{m,n+1,r-1}. \tag{5.12}$$

In comparison with the biHermite case, we need to use a new relation to substitute for the lack of infinitely many initial conditions. We get this relation by applying $R(Q_{m+1,n,r-1}, \cdot)$, $R(Q_{m-1,n+1,r}, \cdot)$ and $R(Q_{m,n-1,r+1})$ in turn to (5.6), and taking the product of these three relations so that unwanted factors drop out. The result, after the substantial straightforward cancellation, is

$$\begin{aligned} &R_{m,n-1,r+1}^{m-1,n,r+1} R_{m-1,n+1,r}^{m,n+1,r-1} R_{m+1,n,r-1}^{m+1,n-1,r} = \\ &\qquad R_{m+1,n-1,r}^{m,n-1,r+1} R_{m-1,n,r+1}^{m-1,n+1,r} R_{m,n+1,r-1}^{m+1,n,r-1} \cdot \\ &\qquad (3m-1)^{3m-1}(3n-1)^{3n-1}(3r-1)^{3r-1}. \end{aligned} \tag{5.13}$$

If one replaces the D's and R's in (5.11), (5.12), and (5.13) by f's using the not-yet-known formulas (5.9) and (5.10) one gets relations among the f's which need to be checked. For example, the f-version of (5.11) is

$$f_m^m f_m^m f_n^n f_n^n f_r^r f_m^m = f_{m+1}^m f_n^n f_r^{r-1} f_m^{m-1} f_n^n f_r^{r+1}.$$

This equation is true because of (5.7) applied twice, once with $x = m$ and once with $x = r$. Similarly, the f-version of (5.12) is true because of (5.7)

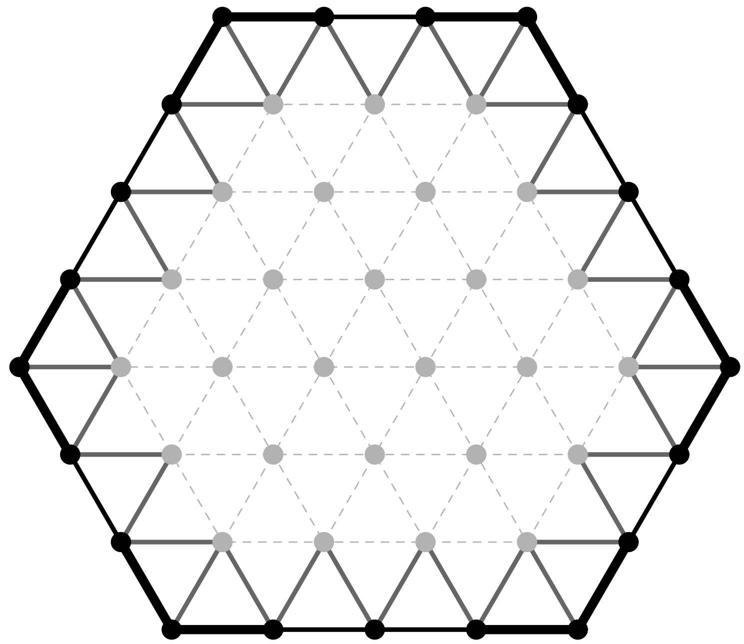

FIGURE 5.1. Indices for the inductive step $G_3 \to G_4$. Vertices index discriminants and edges index resultants.

applied with $x = n$. Finally, the f-version of (5.13) is true because of three applications of (5.8), one each to $x = m$, n, and r.

It is best to present the inductive aspects of our argument geometrically. We will illustrate them by Figure 5.1. View m, n, and r as coordinate functions on the plane of Figure 5.1, satisfying $m + n + r = 1$. The central point in this figure is the non-lattice point $(1/3, 1/3, 1/3)$. If (m, n, r) is a lattice point then we call the sum of the positive numbers among m, n, and r its height. So there are three points of height one, those with $\{m, n, r\} = \{1, 0, 0\}$. There are nine points of height two, three with $\{m, n, r\} = \{1, 1, -1\}$ and six with $\{m, n, r\} = \{2, 0, -1\}$. In general, there are $6h - 3$ lattice points of height h. Because of the S_3 symmetry, we do not need to indicate which collections of parallel lines correspond to constant integral values of which variable.

Let G_h be the graph with vertices the lattice points of height $\leq h$ and edges connecting adjacent vertices. So Figure 5.1 shows all of G_3 in gray, with edges dashed. It shows the rest of G_4 in gray and black, edges not being dashed. The union of the G_h is an infinite planar graph we call G.

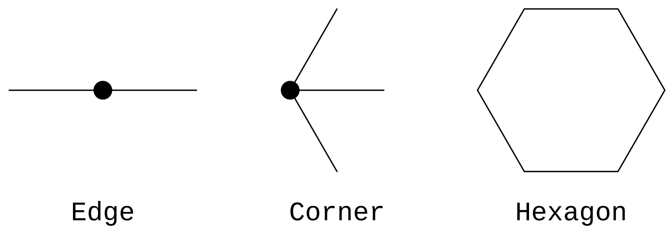

FIGURE 5.2. Diagrams indicating factors present in Equations (5.11), (5.12), (5.13), respectively.

To the vertex $(m, n, r) \in G$ there corresponds the real number $D_{m,n,r}$. To the edge in G connecting (m, n, r) with the adjacent vertex (m', n', r') corresponds the real number $R_{m,n,r}^{m',n',r'}$. Note that the two functions

$$\begin{aligned} D : \text{Vertices} &\rightarrow \mathbf{R}_{\geq 0} \\ R : \text{Edges} &\rightarrow \mathbf{R}_{\geq 0} \end{aligned}$$

are S_3-invariant. So are the formulas on the right side of (5.9) and (5.10). The diagrams in Figure 5.2 correspond to relations (5.11), (5.12), and (5.13) respectively. Consider the edge graph rigidly embedded somehow into G; "rigidly" means here "no bending in the middle"; however different orientations are allowed, according to the S_3 symmetry. One has a discriminant corresponding to the middle vertex and two resultants corresponding to the two edges. If our formula is true for two of these quantities then it is true for the third. Similarly, consider the corner graph rigidly embedded into G. Here one has a discriminant and three resultants. If our formula is true for three of these quantities it is true for the fourth. Finally, consider the hexagon graph rigidly embedded. Here one has six resultants corresponding to the six edges. If our formula is true for five of these six resultants it is true for the last.

We begin the induction at G_3, for which the truth of the formulas can be checked by direct computation. Next we use the edge relation to extend the domain on which the formula is known to the edges which have a vertex in G_3; these 36 edges are drawn in gray. Next we use the hexagon relation to extend this checked domain to the remaining edges except the twelve remaining edges which are incident upon one of the six corners; these 9 edges are drawn as thin black lines. Third, all vertices except the six corners are obtained by using corner relations. Fourth, the twelve remaining edges are obtained by edge relations. Fifth, the six vertices are obtained by corner relations. These last three parts of G_4 are all shown in black, with thick edges. The remaining inductive steps $G_n \to G_{n+1}$ are done in the same way. □

6. Larger picture

It is known that Painlevé I has no rational solutions. It is also known that the rational functions discussed in §1 are the only rational solutions to Painlevé II and Painlevé IV. Finally, there are complete lists of rational solutions to Painlevé III, V, VI, coming from polynomials analogous to P_m, $H_{m,n}$, and $Q_{m,n}$. The extreme case of Painlevé VI is particularly interesting; see [NOOU] for explicit polynomials and [Ma] for the completeness result.

The Painlevé polynomials relating to Painlevé III, V, and VI fit into continuous families. They can be divided into two types:

- Deformations of Yablonsky-Voboriev polynomials. We call these *triangular* Painlevé polynomials.
- Deformations of biHermite polynomials. We call these *biclassical* polynomials because the case $n = 1$ gives the classical Hermite-Laguerre-Jacobi hierarchy.

We have computationally investigated all these Painlevé polynomials in considerable detail. For example, from our computations we have extracted conjectural discriminant formulas for the remaining Painlevé polynomials. Presumably these discriminant formulas have proofs similar to the three presented here. The discriminant formulas in the biclassical case reduce for $n = 1$ to classical formulas dating back to Hilbert [Hi] and Stieltjes [St].

We are quite convinced that all these discriminant formulas form just part of a larger picture. Here are very brief sketches of topics we plan to treat more fully in the future.

Complex roots of P_m, $H_{m,n}$, and $Q_{m,n}$. In Figure 6.1, each of the four squares represents the region $|\mathrm{Re}(x)|, |\mathrm{Im}(x)| \leq 5$ in the complex x-plane. In each square, we have plotted the roots of the indicated Painlevé polynomial. In general, one has the following experimental observations. The roots of P_m approximately form a triangle with m roots on a side. The roots of $H_{m,n}$ approximately form an $m \times n$ rectangle for $m/n \sim 1$; if m/n is far from 1 then the rectangle becomes quite distorted but the rectangular structure remains clearly visible. For $m, n \geq 0$, the roots of $Q_{m,n}$ and $Q_{-n,-m}$ approximately form an $m \times n$ rectangle appropriately framed by triangles in accordance with the degree formulas

$$\mathrm{degree}(Q_{m,n}) = mn + 2\Delta_{m-1} + 2\Delta_{n-1} \tag{6.1}$$

$$\mathrm{degree}(Q_{-n,-m}) = mn + 2\Delta_m + 2\Delta_n. \tag{6.2}$$

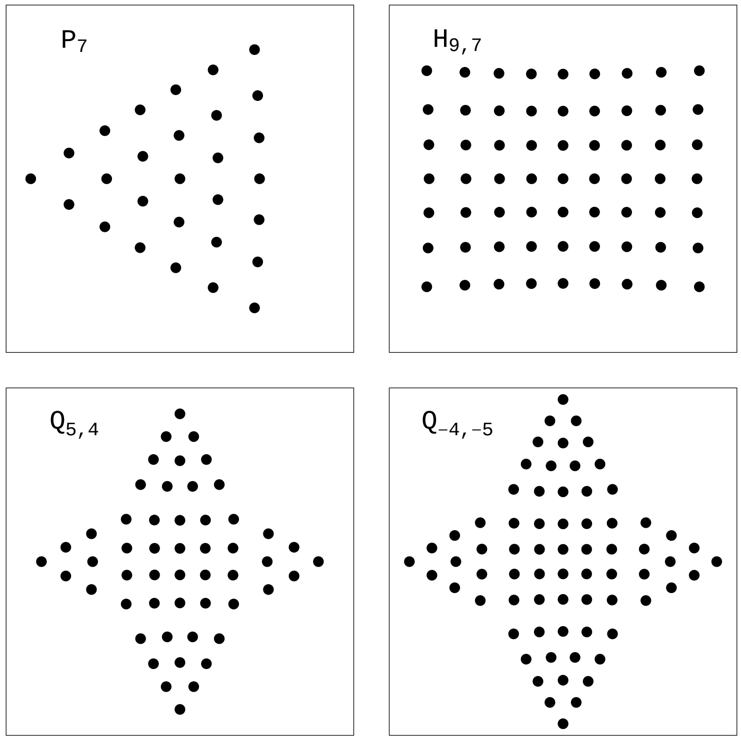

FIGURE 6.1. Complex roots of four Painlevé polynomials

Here both (6.1) and (6.2) are rewritten versions of the symmetric degree formula (5.1). In all three cases the roots behave so regularly that, after separating out polynomials according to their indices modulo 3, 2, and 2, respectively, one can interpolate roots with m and $n \in \mathbf{R}$.

p-adic roots of P_m, $H_{m,n}$, and $Q_{m,n}$. Our discriminant formulas say that 2 doesn't divide $D(P_m)$ and 3 doesn't divide $D(Q_{m,n})$. The separable polynomials $P_m \in \mathbf{F}_2[x]$ and $Q_{m,n} \in \mathbf{F}_3[x]$ factor in an unusually regular way.

In all other cases, one can use Newton polygons to investigate the ramification at p. These polygons behave very regularly. Combining Newton polygon information with our discriminant formulas gives prime-by-prime bounds on the algebra discriminants of $\mathbf{Q}[x]/P_m(x)$, $\mathbf{Q}[x]/H_{m,n}(x)$, and $\mathbf{Q}[x]/Q_{m,n}(x)$. The simplest behavior is for large primes dividing the polynomial discriminant, namely the primes which contribute to only one factor in the polynomial discriminant formula. The p-adic roots exhibit sufficient periodicity that it makes sense to interpolate them with m and n in $\mathbf{Z}_p$.

Complex monodromy of Painlevé III, V, and VI polynomials. As mentioned above, these remaining Painlevé polynomials depend on free parameters. Geometrically, the new parameters let complex roots as in the top two frames of Figure 6.1 move. Our conjectural discriminant formulas identify the locus D in the complex parameter space P where complex roots come together. Computation suggests extremely regular behavior for how roots are interchanged as one moves the complex parameters through $P - D$.

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