Hypergeometric motives and their reduction modulo ℓ

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1. Generalities

The thrice-punctured line

Hypergeometric motives

Reduction modulo ℓ

2. Examples

$H(\frac{1}{2}, \ \ldots, \ \frac{1}{2}; 0, \ \ldots, \ 0; \mathbb{Z})_t$

$H(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 0, 0, 0, 0, 0; \mathbb{F}_3)_t$ *via polys*

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1. Generalities. *The thrice-punctured line.*

Let $T^{\mathrm{an}} = \mathbb{C} - \{0, 1\}$, base-pointed at $\star = 1/2$. Then

$$\begin{aligned} \pi_1(T^{\mathrm{an}}, \star) &= \langle g_0, g_1 \rangle \\ &= \langle g_0, g_1, g_\infty | g_0 g_1 g_\infty = 1 \rangle. \end{aligned}$$

One can also consider the thrice-punctured line T over $\mathbb{Q}$, with

$$\widehat{\pi}_1(T^{\mathrm{an}}, \star) \hookrightarrow \widehat{\pi}_1(T, \star) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

We are interested in representations of the profinite group $\widehat{\pi}_1(T, \star)$ into $GL_n(E)$ for various rings E. They have both a geometric and arithmetic nature.

Hypergeometric motives.

Let $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d \in \mathbb{Q}/\mathbb{Z}$, with common denominator c.

Let $a_j = e^{2\pi i \alpha_j}$ and $b_j = e^{2\pi i \beta_j}$.

Let

$$\begin{aligned} f(x) &= x^d + A_1 x^{d-1} + \cdots \\ g(x) &= x^d + B_1 x^{d-1} + \cdots \end{aligned}$$

have roots a_j and b_j respectively.

Let h_∞^{-1} and h_0 be the corresponding companion matrices. Then $\langle h_0, h_\infty \rangle$ acts irreducibly on $\mathbb{C}^d$ if and only if $\alpha_i \neq \beta_j$ always.

Assume (simplest case that we're focusing on here) that the α's and β's are stable under multiplication by elements of $(\mathbb{Z}/c)^\times$. Then both h_0 and h_∞ are in $GL_d(\mathbb{Z})$.

The data determine a rank d local system $H(\alpha;\beta;\mathbb{Z})$ over T^{an}. Thus for every $t \in T^{an}$, one has a rank d free $\mathbb{Z}$-module $H(\alpha,\beta,\mathbb{Z})_t$. The monodromy about 0, 1, and ∞ is given by h_0, $h_1 = h_0^{-1}h_\infty^{-1}$, and h_∞.

The richness of the situation comes because $H(\alpha;\beta;\mathbb{Z})$ is *motivic*. This means, among other things, that for $t \in T(\mathbb{Q}) = \mathbb{Q} - \{0,1\}$, the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H(\alpha;\beta;\mathbb{Z}_\ell)_t$ and as ℓ varies these actions are "compatible" with each other.

A prime p can be ramified in $H(\alpha;\beta;\mathbb{Z}_\ell)_t$ only if p divides

$$c \cdot \ell \cdot \mathrm{num}(t) \cdot \mathrm{denom}(t) \cdot \mathrm{num}(t-1).$$

It is at worst tamely ramified unless p divides $c \cdot \ell$.

Reductions modulo ℓ. Even the reductions $H(\alpha;\beta;\mathbb{F}_\ell)$ are interesting. Two interesting phenomena occur:

1) $H(\alpha;\beta;\mathbb{F}_\ell)$ depends only on the prime-to-ℓ parts $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β.

2) $\langle h_0, h_\infty \rangle$ acts irreducibly on $\overline{\mathbb{F}}_\ell^d$ if and only if $\widehat{\alpha}_i \neq \widehat{\beta}_j$ always.

2. Examples. A sequence of examples is

$$H_d(E) = H(\overbrace{\frac{1}{2}, \ldots, \frac{1}{2}}^{d}; \overbrace{0, \ldots, 0}^{d}; E).$$

Then, with $X_t: \; y^2 = x(x-1)(x-t)$,

$$\begin{aligned} H_1(E)_t &= E\sqrt{t(1-t)} \\ H_2(E)_t &= H^1(X_t, E) \\ H_3(E)_t &= \mathrm{Sym}^2 H^1(X_t, E) \otimes E\sqrt{t(1-t)} \end{aligned}$$

In general the Zariski closure of $\langle h_0, h_\infty \rangle$ in $GL_d(E)$ is small if char$(E) = 2$ and otherwise

$$\begin{cases} Sp_d(E) & \text{if } d \text{ is even} \\ O_d(E) & \text{if } d \text{ is odd} \end{cases}$$

The choice $E = \mathbb{F}_3$ is interesting here. We'll pursue the case $d = 5$. Here $O_5(\mathbb{F}_3)/(\pm 1)$ has the form $H.2$ where H is a simple group of order $25920 = 2^6 3^4 5$. The action of $H.2$ on $\mathbb{P}^4(\mathbb{F}_3)$ has orbit-decomposition

$$|\mathbb{P}^4(\mathbb{F}_3)| = \frac{3^5 - 1}{3 - 1} = 121 = 36 + 40b + 45.$$

$$f_{27}(u,x) = 2^{12}\left(3x^3 - 3x - 1\right)^9$$
$$-u\left(48x^3 + 108x^2 + 63x + 11\right) \cdot$$
$$\left(36x^6 + 162x^5 + 135x^4 + 138x^3 + 108x^2 + 36x + 4\right)^4$$

$$f_{36}(u,x) = 2^{12}3^3\left(x^3 - 3x + 1\right)^9$$
$$-u\left(x^6 - 3x^5 - 30x^4 - 19x^3 + 78x^2 + 33x - 83\right)^2 \cdot$$
$$\left(x^6 - 3x^5 - 3x^4 + 8x^3 - 3x^2 - 21x + 25\right)^4$$

$$f_{40a}(u,x) = 2^{12}(3x-1)\left(3x^3 - 3x + 1\right)^9\left(3x^3 - 9x^2 + 6x - 1\right)^3$$
$$-u\left(3x^2 - 6x + 2\right)^2\left(9x^4 + 24x^3 - 24x + 8\right) \cdot$$
$$\left(27x^8 - 54x^7 - 36x^6 + 504x^5 - 846x^4 + 624x^3 - 240x^2 + 4\right.$$

$$f_{40b}(u,x) = 2^{12}3^3(x-2)^3(x+1)\left(x^3 - 3x + 1\right)^9$$
$$-u\left(x^6 - 3x^5 + 6x^4 - 19x^3 + 24x^2 + 15x - 29\right)^4 \cdot$$
$$\left(x^8 + 8x^7 + 10x^6 - 64x^5 - 125x^4 + 224x^3 + 214x^2 - 88x -\right.$$

$$f_{45}(u,x) = 2^{12}3^3\left(x^3 - 3x + 1\right)^9\left(x^3 + 3x^2 - 6x + 1\right)^3$$
$$-u(x-2)\left(x^6 + 6x^5 + 6x^4 - 10x^3 - 75x^2 + 96x - 20\right)^2$$
$$\left(x^8 + 2x^7 - 14x^6 - 10x^5 + 70x^4 - 130x^3 + 106x^2 - 22x + 1\right.$$

All with $\pm 2^* 3^* u^* (u-1)^*$. Use $u = \frac{-4t}{(t-1)^2}$.

3A. Tame Ramification. Primes not dividing $c{\cdot}\ell$ are tame and behave very simply:

If t is p-adically k-close to the cusp $c \in \{0, 1, \infty\}$, the inertial element τ_p is the class h_c^k.

Illustration. Take $t = 50$. Then $t = 2 \cdot 5^2$ is 5-adically 2-close to 0. The algebras $\mathbb{Q}_5[x]/f_N(50, x)$ factor over $\mathbb{Q}_5$ as follows:

27	9^3
36	$9^3 \times 9$
$40a$	$9^3 \times 3^3 \times 3 \times 1$
$40b$	$9^3 \times 9 \times 3 \times 1$
45	$9^3 \times 9 \times 3^3$.

The regular unipotent nature of $\tau_5 = h_0^2$ is evident from all the 9's.

Also $t = 1 + 7^2$ is 7-adically 2-close to 1. The fact that $\tau_7 = h_1^2 =$ Identity is evident from the fields $\mathbb{Q}[x]/f_N(50, x)$ all being unramified at 7.

3B. Wild Ramification. Primes dividing $c \cdot \ell$ are generally wild and behave in a very complicated way. However wild ramification can be investigated in $H(\alpha; \beta; \mathbb{F}_\ell)_t$ via equations and this sheds light on $H(\alpha; \beta; \mathbb{Z})_t$.

Illustration. The next two slides gives 2-adic and 3-adic ramification in

$$H(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 0, 0, 0, 0, 0; \mathbb{F}_3)_t$$

as a function of t:

t	e	27	36	$40a$	$40b$	45	wild slopes				e	f	grd
0	1	1				1	3.5	3	2	2	1	4	7.34
0	2_{001}	3				1		2	2	2	1	4	3.36
0	2_{11}	1				1	2	2	2	2	1	4	3.67
0	2_{101}	1				1		2	2	2	1	4	3.36
0	3	1					2.8	2.8	2.8	2.8	5	4	10.73
0	4_{01}	2	1								5	4	1.74
0	4_{11}	1					1.6	1.6	1.6	1.6	5	4	4.92
0	5	1					2.4	2.4	2.4	2.4	5	4	8.28
0	6	1					1.6	1.6	1.6	1.6	5	4	4.92
0	7	1				1	2	2	2	2	1	4	3.67
0	8	1					1.2	1.2	1.2	1.2	5	4	2.26
0	9	1					1.6	1.6	1.6	1.6	5	4	4.92
0	10	1					1.2	1.2	1.2	1.2	5	4	2.26
0	11	1					1.2	1.2	1.2	1.2	5	4	2.26
0	12		1								1	10	1.00
0	$13+3k$			1	1						9	6	1.85
0	$14+3k$			1	1						9	6	1.85
0	$15+9k$			2	2						3	6	1.59
0	$18+9k$			2	2						3	6	1.59
0	$21+9k$		1	2	3	1					1	6	1.00
1	1	3				1	3.5	3	2	2	1	2	7.34
1	2_{01}	3				3			3	2	1	2	4.00
1	2_{11}	3				1			3	2	1	4	4.00
1	$3+2k$	3				1		3	2	2	1	2	4.76
1	$4+2k$	3				1			3	2	1	4	4.00
∞	1_{01}	1		2		1			4	3	1	2	6.73
∞	1_{11}	1				1			4	3	1	4	6.73
∞	$(2+4k)_{01}$	1				1		3	2	2	1	4	4.76
∞	$(2+4k)_{11}$	1				1		2	2	2	1	4	3.36
∞	$3+2k$	1				1		4	3	2	1	4	8.00
∞	$(4+4k)_{01}$	1				1		2	2	2	1	4	3.36
∞	$(4+4k)_{11}$	1				1		3	2	2	1	4	4.76

t	e	27	36	$40a$	$40b$	45	wild slopes				e	f	grd
g	0			1	1			2.17	2	1.5	2	1	9.18
g	$10, 12$			1	2				2.5	1.5	2	3	9.57
g	11			4	2				2.5	1.5	2	1	9.57
g	2			2	1			2.5	2	1.5	2	1	11.72
0	1			1	1		3.5	2.67	2.5	1.5	2		31.99
0	2_1			1	1			3.5	2.5	2	2		28.70
0	2_{*2}			1	1				3.5	2.5	2	3	25.40
0	2_{22}			1	1				3.5	2.5	2	1	25.40
0	$(3+3k)^e_{01}$		1	2	3	1				2	2	1	5.20
0	$(3+3k)^g_{01}$			2	2				2	1.5	2		6.63
0	$(3+3k)_{*1}$			1	1			2.5	2	1.5	2		11.72
0	$(3+3k)_{*2}$			1	1			2.5	2	1.5	2		11.72
0	$(3+3k)^g_{22}$			2	2				2	1.5	2		6.63
0	$(3+3k)^e_{22}$		1	2	3	1				2	2	1	5.20
0	$4+3k$			1	1			3.5	2.5	2	2		28.70
0	$5+3k$			1	1			3.5	2.5	2	2		28.70
1	1			1	1		2.83	2.67	2.5	1.5	2		19.63
1	2_1				1			3.5	2.67	2.5	2		31.13
1	2_2		1		1				2.25	2.25	4	2	9.86
1	3_1	1			1					2	4	2	5.69
1	3_2				1			2	1.25	1.25	4	2	6.70
1	4				1			2	1.25	1.25	4	2	6.70
1	$5+k$	1			1					2	4	2	5.69
∞	1			1	1		$3.1\overline{6}$	$2.\overline{6}$	2.5	1.5	2		25.06
∞	2			1				$2.8\overline{3}$	2.375	2.375	8	2	17.88
∞	3_{11}			1		1				2.5	8	2	8.60
∞	3_{*1}			1				2.5	1.125	1.125	8	2	9.33
∞	3_{12}			1		1				2.5	8	2	8.60
∞	3_{*2}			1				2.5	1.125	1.125	8	2	9.33
∞	4			1				2.5	1.875	1.875	8	2	11.91
∞	5			1				2.5	1.625	1.625	8	2	10.97
∞	6			1				2.5	1.125	1.125	8	2	9.33
∞	7			1				2.5	1.125	1.125	8	2	9.33
∞	$8+k$			1		1				2.5	8	2	8.60

4A. Rank four symplectic motives mod 3 and degree 80 even polynomials. There are 13 rank four symplectic hypergeometric motives with coefficients in $\mathbb{F}_3$:

BH	α	β	wild		
24	$(0,0,0,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2	3	
25	$(0,0,\frac{1}{4},\frac{3}{4})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2	3	
26	$(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2	3	
27	$(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8})$	$(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4})$	2	3	
28	$(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2	3	
29	$(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$	$(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$	2	3	5
30	“	$(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$	2	3	5
31	“	$(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8})$	2	3	5
32	“	$(0,0,\frac{1}{2},\frac{1}{2})$	2	3	5
33	“	$(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4})$	2	3	5
34	“	$(0,0,\frac{1}{4},\frac{3}{4})$	2	3	5
35	“	$(0,0,0,0)$		3	5
36	“	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2	3	5

They all lift to finite monodromy motives with coefficients in $\mathbb{Q}(\sqrt{-3})$. on the Beukers-Heckman list.

There is an explicit degree 80 polynomial

$$f(b,c,d,e;x) = x^{80} + 15120bx^{76} + \cdots$$

having 1673 terms and giving three-torsion in

$$y^2 = x^5 + bx^3 + cx^2 + dx + e.$$

By specializing f, at least nine of the thirteen mod 3 representations are identified:

BH	b	c	d	e
24				
25	$-t$	t	0	$-t^2$
26	$6t$	$16t$	$9t^2$	0
27	$-4t$	$4t$	$-t$	0
28	0	$4t$	$3t$	0
29				
30	$-10t$	0	$25t^2$	$16t^2$
31	0	0	$-5t$	$4t$
32				
33	$-15t$	0	0	$162t^2$
34				
35	0	$40t$	$-60t$	$144t$
36	0	$-20t$	0	$48t^2$

4B. Rank eight orthogonal motives mod 2 and degree 240 even polynomials. The last fifteen entries on the Beukers-Heckman list, BH63-BH77, have Galois group

$$W(E_8) = 2.O_8^+(2).2$$

of order $2^{14}3^5 5^2 7 = 696,729,600$. The last one, BH77, is

$$\begin{aligned}
\alpha &= \left(\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}, \frac{13}{24}, \frac{17}{24}, \frac{19}{24}, \frac{23}{24}\right) \\
\beta &= \left(0, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\right) \quad \text{with} \\
\widehat{\alpha} &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \\
\widehat{\beta} &= \left(0, 0, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\right).
\end{aligned}$$

A polynomial for it comes from specializing Shioda's generic $W(E_8)$ polynomial:

$$\begin{gathered}
S(0,0,0,0,54t,-243t,-729t,0) = \\
x^{240} + 141523200tx^{222} + \cdots
\end{gathered}$$